Continuous percolation phase transitions of random networks under a generalized Achlioptas process

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Using finite-size scaling, we have investigated the percolation phase transitions of evolving random networks under a generalized Achlioptas process (GAP). During this GAP, the edge with a minimum product of two connecting cluster sizes is taken with a probability \( p \) from two randomly chosen edges. This model becomes the Erdős-Rényi network at \( p = 0.5 \) and the random network under the Achlioptas process at \( p = 1 \). Using both the fixed point of the size ratio \( s_2/s_1 \) and the straight line of \( \ln s_1 \), where \( s_1 \) and \( s_2 \) are the reduced sizes of the largest and the second-largest cluster, we demonstrate that the phase transitions of this model are continuous for \( 0.5 \leq p \leq 1 \). From the slopes of \( \ln s_1 \) and \( \ln(s_2/s_1) \) at the critical point, we get critical exponents \( \beta \) and \( \nu \) of the phase transitions. At \( 0.5 \leq p \leq 0.8 \), it is found that \( \beta, \nu \), and \( s_2/s_1 \) at critical point are unchanged and the phase transitions belong to the same universality class. When \( p \geq 0.9 \), \( \beta, \nu \), and \( s_2/s_1 \) at critical point vary with \( p \) and the universality class of phase transitions depends on \( p \).

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I. INTRODUCTION

The modern theory of complex networks [1–3] has opened new perspectives in the study of complex systems in nature, social and economic systems, technical infrastructures, and many other fields. The macroscopic properties of complex networks emerge from the interactions among individual constituents. The percolation phase transition of complex networks is an interesting macroscopic property and can be studied with the percolation theory in statistical physics [4]. It has been pointed out that the random network undergoes a continuous percolation phase transition during the random process [5]. The critical phenomena of complex networks are reviewed in Ref. [6].

However, Achlioptas et al. [7] reported that the percolation phase transition of random networks becomes discontinuous under the Achlioptas process (AP). During this AP, two unoccupied edges are chosen randomly and the edge with minimum product of the connecting cluster sizes is taken as the next occupied bond. This discontinuous phase transition in networks is called the explosive percolation [7]. The AP was applied later to other networks [8–10] and some similar rules were also introduced [9–15]. Using the finite-size scaling, the explosive percolation phase transition was also investigated [16–18]. These works support that the explosive percolation is discontinuous. Recently, Costa et al. [19] showed that the explosive percolation transition is actually a continuous phase transition with a uniquely small critical exponent. Riordan and Warnke [20] prove mathematically that all Achlioptas processes have continuous phase transitions.

In this paper we investigate the percolation phase transitions of random networks under a generalized Achlioptas process (GAP). In the GAP, two unoccupied edges are chosen randomly and the edge with minimum product of two connecting cluster sizes is taken with a probability \( p \) [21,22]. At \( p = 0.5 \), this model becomes the Erdős-Rényi (ER) network [23]. At \( p = 1 \), the GAP is equivalent to the Achlioptas process, which suppresses the appearance of larger clusters. With the increase of \( p \) from 0.5 to 1, this suppression is switched on gradually. After investigating the percolation phase transitions at different \( p \), we can get more understanding about these phase transitions and make a more reliable conclusion about the character of the percolation phase transition at \( p = 1 \).

II. FINITE-SIZE SCALING OF CLUSTERS

In our Monte Carlo simulations, we begin with \( N \) isolated nodes and then connect them with the edges added through the GAP. The network obtained is characterized by \( N \), the number of edges \( N_r \) and the probability parameter \( p \) of GAP. For \( p < 1 \), we take \( N = (4,8,16,32,64,128) \times 10^4 \) and run 500,000 steps in each simulation. At \( p = 1 \), we choose much larger system sizes \( N = (512,1024,2048) \times 10^4 \) to reach the asymptotic region of finite-size effects and run 100,000 steps in each simulation.

For the cluster ranked \( R \) and with size \( S_R(N_r,N;p) \), we define its reduced size as

\[
s_R(r,N;p) \equiv S_R(N_r,N;p)/N,\tag{1}
\]

where \( r \equiv N_r/N \) is the reduced edge number. The reduced size of the largest cluster \( s_1(r,N;p) \) at \( N = 1.28 \times 10^6 \) is shown in Fig. 1. \( s_1 \) becomes finite for \( r > r_c \) and there is a percolation phase transition. The transition point \( r_c \) increases with \( p \) and the corresponding phase transition becomes sharper.

If the percolation phase transition is continuous, the reduced sizes \( s_R(r,N;p) \) should follow a finite-size scaling form [24,25]

\[
s_R(r,N;p) = N^{-\beta/\nu} \tilde{s}_R(t(N^{1/\nu};p),\tag{2}
\]
FIG. 1. (Color online) Reduced sizes of the largest cluster as a function of the reduced edge number $r$ in the network with node number $N = 1.28 \times 10^6$ and different probability parameter $p$. where $t = (r - r_c)/r_c$ and $\nu$ is the critical exponent of the correlation length $\xi = \xi_0 |t|^{-\nu}$. The scaling form in Eq. (2)

$$s_2/s_1 = \tilde{s}_2(tN^{1/\nu}; p)/\tilde{s}_1(tN^{1/\nu}; p) \equiv U(tN^{1/\nu}; p)$$

is valid in the asymptotic critical region with $|t| \ll 1$ and $N \gg 1$. Outside this region, additional correction terms should be taken into account. This finite-size scaling form originates from the long-range correlation near a critical point. When the correlation distance in a system becomes comparable with its size, the macroscopic properties are determined only by the ratio of the system size to the correlation length, which can be expressed also as $tN^{1/\nu}$. At $r = r_c$, both $\tilde{s}_1$ and $\tilde{s}_2$ are constant with respect to the change of system size $N$. When $r < r_c$, $\tilde{s}_1$ and $\tilde{s}_2$ decrease with the increase of $N$. For $r > r_c$, $\tilde{s}_1$ increases while $\tilde{s}_2$ decreases with $N$. In Ref. [26], the finite-size scaling functions $\tilde{s}_1$ and $\tilde{s}_2$ of two-dimensional site percolation are calculated with the Monte Carlo simulations of lattice sizes from $L = 50$ to $4000$.

From Eq. (2), we can obtain the size ratio

$$s_2/s_1 = \tilde{s}_2(tN^{1/\nu}; p)/\tilde{s}_1(tN^{1/\nu}; p) \equiv U(tN^{1/\nu}; p)$$

and

$$\ln s_1(r, N; p) = -\beta/\nu \ln N + \ln \tilde{s}_1(tN^{1/\nu}; p).$$

FIG. 2. (Color online) (a) Size ratio $s_2/s_1$ for $p = 0.5$ has a fixed point at $r_c^1 = 0.5000 \pm 0.0004$; (b) Log-log plot of the reduced size $s_1$ versus $N$. Around $r_c^1 = 0.5000$, the curvature of $\ln s_1$ changes its sign. From the slope of $\ln s_1$ with respect to $\ln N$ at $r_c^1$, we obtain the critical exponent ratio $\beta/\nu = 0.33(1)$ at $p = 0.5$.

FIG. 3. (Color online) (a) Size ratio $s_2/s_1$ for $p = 0.8$ has a fixed point at $r_c^1 = 0.7252 \pm 0.0001$; (b) Log-log plot of the reduced size $s_1$ versus $N$ at different $r$. Around $r_c^1 = 0.7252$, the curvature of $\ln s_1$ changes its sign. From the slope of $\ln s_1$ with respect to $\ln N$ at $r_c^1$, we obtain the critical exponent ratio $\beta/\nu = 0.33(1)$ at $p = 0.8$. 

061110-2
At a critical point, \( s_2 / s_1 \rangle \left. \right|_{\text{r-crit}} = U(0; p) \) is a fixed point versus \( N \) and \( \ln s_1(\nu_c, N; p) = \beta / \nu \ln N + \ln s_1(0; p) \) is a straight line versus \( \ln N \). Using these properties, the critical point of complex network can be determined both from the fixed point of \( s_2 / s_1 \) and the straight line of \( \ln s_1 \) versus \( \ln N \). We denote the critical point obtained from \( s_2 / s_1 \) as \( r_c^2 \). The critical point determined from \( \ln s_1 \) is denoted as \( r_c^1 \). The critical exponent ratio \( \beta / \nu \) can be obtained from the slope of \( \ln s_1 \) at \( r_c^2 \).

III. RESULTS

For \( p = 0.5 \), our model is equivalent to the ER model [23]. Its continuous percolation phase transition can be studied with \( s_2 / s_1 \) and \( \ln s_1 \). In Fig. 2(a), \( s_2 / s_1 \) is shown as a function of \( r \) at different \( N \). The different curves of \( s_2 / s_1 \) for different \( N \) have a fixed point at \( r_c^1 = 0.5000(4) \). In Fig. 2(b), \( \ln s_1 \) is shown versus \( \ln N \) for different \( r \). The curvature of \( \ln s_1 \) is negative at \( r = 0.4996 \) and becomes positive at \( r = 0.5004 \). At \( r = 0.5 \), \( \ln s_1 \) is a straight line with slope \(-0.33(1)\). We get \( r_c^2 = 0.5000(4) \), consistent with \( r_c^1 \). Our results of \( r_c \) and \( \beta / \nu \) agree with the exact results \( r_c = 0.5 \) and \( \beta / \nu = 1 / 3 \) of the ER model [23].

In Fig. 3, we show \( s_2 / s_1 \) and \( \ln s_1 \) of a random network under the GAP at \( p = 0.8 \). From the fixed point of \( s_2 / s_1 \), we get the critical point \( r_c^1 = 0.7252 \pm 0.0001 \). The curvature of \( \ln s_1 \) is negative at \( r = 0.7250 \) and becomes positive at \( r = 0.7254 \). So \( r_c^2 = 0.7252 \pm 0.0002 \), which is consistent with \( r_c^1 \).

For \( p = 1.0 \), our model becomes the PR model of Ref. [7].\( s_2 / s_1 \) at different \( N \) are shown in Fig. 4(a) and there is a fixed point at \( r_c^1 = 0.88844(2) \). The curvature of \( \ln s_1 \) is negative at \( r = 0.88840 \) and becomes positive at \( r = 0.88850 \). We obtain \( r_c^2 = 0.88845(5) \), consistent with \( r_c^1 \). From the slope of \( \ln s_1 \) at \( r_c^2 \), we get \( \beta / \nu = 0.04(1) \). Our \( r_c \) and \( \beta / \nu \) at \( p = 1 \) agree well with Ref. [27].

FIG. 4. (Color online) (a) Size ratio \( s_2 / s_1 \) for \( p = 1.0 \) has a fixed point at \( r_c^1 = 0.88844 \pm 0.00002 \). (b) Log-log plot of the reduced size \( s_1 \) versus \( N \). Around \( r_c^2 = 0.88845 \), the curvature of \( \ln s_1 \) changes its sign. From the slope of \( \ln s_1 \) with respect to \( \ln N \) at \( r_c^2 \), we obtain the critical exponent ratio \( \beta / \nu = 0.04(1) \) at \( p = 1.0 \).

FIG. 5. Log-log plot of the derivative \( (s_2 / s_1)’ \) versus \( N \) at \( p = 0.5 \). Around \( r_c^2 = 0.5000 \), the curvature of \( \ln(s_2 / s_1)’ \) changes its sign. From the slope of \( \ln(s_2 / s_1)’ \) with respect to \( \ln N \) at \( r_c^2 \), we obtain the inverse of critical exponent \( 1/\nu = 0.33(1) \) at \( p = 0.5 \).

FIG. 6. Log-log plot of the derivative \( (s_2 / s_1)’ \) versus \( N \) at \( p = 0.8 \). Around \( r_c^2 = 0.7252 \), the curvature of \( \ln(s_2 / s_1)’ \) changes its sign. From the slope of \( \ln(s_2 / s_1)’ \) with respect to \( \ln N \) at \( r_c^2 \), we obtain the inverse of critical exponent \( 1/\nu = 0.35(1) \) at \( p = 0.8 \).
FIG. 7. Log-log plot of the derivative \((s_2/s_1)\)' versus \(N\) at \(p = 1.0\). Around \(r_c^2 = 0.88845\), the curvature of \(\ln(s_2/s_1)\)' changes its sign. From the slope of \(\ln(s_2/s_1)\)' with respect to \(\ln N\) at \(r_c^2\), we obtain the inverse of critical exponent \(1/\nu = 0.50(1)\) at \(p = 1.0\).

To determine the critical exponent \(\nu\) of correlation length, we study the derivative \((s_2/s_1)\)' of\(\equiv \partial(s_2/s_1) / \partial r = N^{1/\nu} r_c^{-1} U'(t N^{1/\nu}; p)\). (5)

At a critical point, we have
\[
\ln(s_2/s_1)'|_{r=r_c} = (1/\nu) \ln N + \ln U'(0; p) - \ln r_c,
\]
which is a straight line with slope \(1/\nu\).

In Fig. 5, a log-log plot of the derivative \((s_2/s_1)\)' versus \(N\) at \(p = 0.5\) is shown around \(r_c = 0.5000\). The function \(\ln(s_2/s_1)\)' becomes a straight line at \(r_c\) and its slope gives the inverse of critical exponent \(1/\nu = 0.33(1)\). In Figs. 6 and 7, the results at \(p = 0.8\) and 1.0 are presented respectively. From the slopes of corresponding straight lines, we obtain \(1/\nu = 0.35(1)\) at \(p = 0.8\) and \(1/\nu = 0.50(1)\) at \(p = 1.0\).

The ratio of critical exponent \(\beta/\nu\) and the inverse of critical exponent \(1/\nu\) at different \(p\) are summarized in Table I. For the probability parameters in the range \(0.5 \leq p \leq 0.8\), our model has the same \(\beta/\nu\) and \(1/\nu\) within the error bars of Monte Carlo data and therefore belongs to the same universality class as the ER model. For \(p \geq 0.9\), both \(\beta/\nu\) and \(1/\nu\) deviate from their values of the ER model and vary with \(p\). Therefore, the

\begin{table}[h]
\centering
\begin{tabular}{cccc}
\hline
\(p\) & \(r_c^1\) & \(r_c^2\) & \(\beta/\nu\) & \(1/\nu\) \\
\hline
0.5 & 0.5000(4) & 0.5000(4) & 0.33(1) & 0.33(1) \\
0.6 & 0.5599(1) & 0.5599(3) & 0.32(1) & 0.35(1) \\
0.7 & 0.6349(2) & 0.6349(3) & 0.32(1) & 0.34(1) \\
0.8 & 0.7252(1) & 0.7252(2) & 0.33(1) & 0.35(1) \\
0.9 & 0.8216(1) & 0.8216(3) & 0.31(1) & 0.38(1) \\
0.95 & 0.8603(2) & 0.8604(2) & 0.15(1) & 0.46(1) \\
1.0 & 0.88844(2) & 0.88845(5) & 0.04(1) & 0.50(1) \\
\hline
\end{tabular}
\caption{Critical reduced edge number \(r_c\), critical exponent ratio \(\beta/\nu\) and inverse of critical exponent \(1/\nu\). We obtain \(r_c^1\) from \(s_2/s_1\) and \(r_c^2\) from \(\ln s_1\).}
\end{table}

FIG. 8. (Color online) Finite-size scaling function of size ratio \(U(t N^{1/\nu}; p)\) at different probability parameters. (a) \(p = 0.5\) with \(1/\nu = 0.33\). (b) \(p = 0.8\) with \(1/\nu = 0.35\). (c) \(p = 1.0\) with \(1/\nu = 0.5\).

universal class of our model at \(p \geq 0.9\) is different from the ER model’s and depends on the probability parameter \(p\). Using \(r_c\) and \(1/\nu\) given in Table I, we can define a finite-size scaling variable \(t N^{1/\nu}\) for each \(p\). With this scaling variable, our Monte Carlo simulation data of \(s_2/s_1(r, N; p)\) at a given \(p\) and different \(N\) collapse into a finite-size scaling function.
cluster sizes is added into the network with a probability \( p \) from two randomly chosen edges. This model becomes the ER network [23] at \( p = 0.5 \) and the random network under the Achlioptas process [7] at \( p = 1 \). Using the finite-size scaling of critical phenomena, the percolation phase transitions of this model are studied. From the finite-size scaling forms of the largest cluster size \( s_1(r,N;p) \) and the second largest cluster size \( s_2(r,N;p) \), the fixed point of \( s_2/s_1 \) and the straight line of \( \ln s_1 \) versus \( \ln N \) can be used to determine the critical points.

It has been found that the critical points \( r_c \), determined from \( s_2/s_1 \) and \( \ln s_1 \) are consistent and increase with \( p \). From the slopes of \( \ln s_1 \) and \( \ln(s_2/s_1)^\prime \) at \( r_c \), we can obtain the ratio of critical exponent \( \beta/\nu \) and the inverse of critical exponent \( 1/\nu \) respectively. For \( 0.5 \leq p \leq 0.8 \), both \( \beta/\nu \) and \( 1/\nu \) are unchanged within the error bars of Monte Carlo simulation data and the phase transitions belong to the same universality class as the ER model’s. When \( p \geq 0.9 \), \( \beta/\nu \) deviate from their values of the ER model. \( \beta/\nu \) decreases from 0.31(1) at \( p = 0.9 \) to 0.04(1) at \( p = 1 \). \( 1/\nu \) increases from 0.38(1) at \( p = 0.9 \) to 0.50(1) at \( p = 1 \). The universality class of the phase transitions at \( p \geq 0.9 \) is different from the ER model’s and depends on \( p \). With the scaling variable \( tN^{1/\nu} \), the Monte Carlo data of \( s_2/s_1(r,N;p) \) at a given \( p \) and different \( N \) collapse into a finite-size scaling function \( U(tN^{1/\nu};p) \). As the critical exponents, the size ratio at critical point \( U(0;p) \) is the same at \( 0.5 \leq p \leq 0.8 \) and depend on \( p \geq 0.9 \).

In conclusion, the percolation phase transitions of random networks under the GAP are always continuous for probability parameter \( 0.5 \leq p \leq 1.0 \). At \( 0.5 \leq p \leq 0.8 \), the phase transitions belong to the same universality class. When \( p \geq 0.9 \), the universality class of phase transitions depends on \( p \).

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