A microscopic convexity theorem of level sets
for solutions to elliptic equations

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Abstract We study the microscopic level-set convexity theorem for elliptic equation
$Lu = f(x, u, Du)$, which generalize Korevaar's result in (Korevaar, Commun Part Diff
Eq 15(4):541–556, 1990) by using different expression for the elementary symmetric func-
tions of the principal curvatures of the level surface. We find out that the structure conditions
on equation are as same as conditions in macroscopic level-set convexity results (see e.g.
(Colesanti and Salani, Math Nachr 258:3–15, 2003; Greco, Bound Value Prob 1–15, 2006)).
In a forthcoming paper, we use the same techniques to deal with Hessian type equations.

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1 Introduction

In this paper we give a constant rank theorem of level sets for solutions to equations

$Lu = f(x, u, Du) \quad \text{in} \quad \Omega \subset \mathbb{R}^n. \quad (1.1)$

where $L$ is quasilinear elliptic operator including the Laplacian and the $p$-Laplacian and the
mean curvature operator.

So far as we know there are two important methods to study convexity of level sets
and convexity of solutions questions, which are “microscopic” and “macroscopic” maxi-
mum principles (see e.g. [12]). For level set arguments, Gabriel [8] was the first to study
“macroscopic” maximum principle for $L = \Delta$ and homogeneous $f$ in convex ring. Then
it was extended by Lewis [14] to $p$-Laplace operator. For inhomogeneous cases, especially
the first eigenvalue problem of the Laplacian, Brascamp and Lieb [2] proved that the first

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eigenfunction is log concave and hence the level sets are convex by using parabolic flow method. Caffarelli and Spruck treated more general inhomogeneous Laplace equation in [4]. The other results in this topic see [11] and references therein. Recently Colesanti and Salani [6] and Greco [10] settled the level-set convexity for special case of equation (1.1) in convex ring. And soon it was generalized to nonlinear elliptic Dirichlet problems by Cuoghi and Salani [7] and later by Bianchini–Longinetti–Salani [1].

The microscopic technique for convex solution was discovered in two-dimension by Caffarelli and Friedman [3], and the n-dimensional in [13] for semilinear equations. The key is a so called “constant rank theorem” which has recently been generalized to fully nonlinear equations (see e.g. [5]). We emphasize that microscopic approach is a strong maximum principle. Its advantage is that it can prove strict convexity and also can be transformed to manifolds.

The breakthrough for the analogous level-set constant rank theorem is owing to Korevaar [12], and it is the only progress in this topic to our knowledge. Our main result is the following generalization.

**Theorem 1.1** Let \( u \in C^4(\Omega) \) solve the elliptic equation

\[
Lu = A(u, |Du|) \left( \Delta u - \frac{u_\alpha u_\beta}{|Du|^2} u_{\alpha\beta} \right) + B(u, |Du|) \left( \frac{u_\alpha u_\beta}{|Du|^2} u_{\alpha\beta} \right) = f(x, u, Du)
\]

(1.2)

where \( A, B, f \) are \( C^2 \) function and subscripts refer to partial derivatives, and where we use the summation convention. Suppose that \( u \) has convex level sets (i.e the sets \( \{ x \in \Omega | u(x) \leq c \} \) are locally convex) and that \( Du \neq 0 \). Suppose further that \( A, B, f \) satisfy the structure conditions with \( t = |Du| \) and \( \mu > 0 \)

\[
\left( \frac{A}{B} (u, t) \right)_{tt} \geq 0 \quad \text{and} \quad \mu^3 f \left( x, u, \frac{Du}{\mu} \right) \text{ concave w.r.t} \ (x, \mu).
\]

(1.3)

Then all the level sets of \( u \) have second fundamental forms with (the same) constant rank through out the connected domain \( \Omega \).

**Remark 1.2** When \( f = f(u, |Du|) \), Theorem 1.1 is proved in [12]. When \( L = \Delta \), i.e. \( A = B = 1 \), the condition (1.3) is the same as (1.5) in [10]. While \( f = f(x, u, |Du|) \) and \( L \) is a special operator, such as

- **mean curvature** : \( A = \frac{1}{\sqrt{1 + |Du|^2}}, \quad B = \frac{1}{(1 + |Du|^2)^{\frac{1}{2}}} \)
- **p-Laplacian** : \( A = |Du|^{p-2}, \quad B = (p-1)|Du|^{p-2} \)

The conditions (1.3) are equivalent to those in [6] (Note the definition of level set in [6,10] is \( \{ x \in \Omega | u(x) \geq c \} \)).

**2 Proof of Theorem 1.1**

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with boundary \( \partial \Omega \in C^2 \), having principal curvatures \( \kappa = (\kappa_1, \ldots, \kappa_{n-1}) > 0 \). For \( m = 0, 1, \ldots, n-1 \), we define the \( m \)th mean curvature of \( \partial \Omega \) by

\[
H_m[\partial \Omega] = S_m(\kappa_1, \ldots, \kappa_{n-1}),
\]

(2.1)
where $S_m$ is $m$-th elementary symmetric function, that is for $1 \leq l \leq n$ and $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$,

$$S_l(\lambda) = \sum_{1 \leq i_1 < \cdots < i_l \leq n} \lambda_{i_1} \cdots \lambda_{i_l}.$$ 

We also let $S_0 = 1$ and denote $S_l(\lambda|i) = S_l(\lambda)_{i_l=0}$. For $n \times n$ symmetric matrix $W$, then

$$S_l(W) = \sum_{1 \leq i_1 < \cdots < i_l \leq n} (-1)^{\text{sgn}(\sigma)} W_{i_1}^{i_2} \cdots W_{i_l}^{i_l},$$

Let $\Sigma_c = \{ x \in \Omega \mid u(x) = c \}$ be the connected level surface for any constant $c$, suppose that $u$ has convex level sets ($\{ x \in \Omega \mid u(x) \leq c \}$), and that $Du \neq 0$, that is the second fundamental form of level surface is nonnegative with respect to $-Du$, then according to [15], for any $k = 0, 1, \ldots, n - 1$

$$H_k[\Sigma_c] = \frac{\partial S_{k+1}}{\partial u_{ij}} (D^2u)_{ij} \frac{1}{|Du|^{k+2}} \geq 0.$$ 

(2.2)

Here and below we use the summation convention of summing over repeated indices from 1 to $n$. We will utilize this expression for $H_k[\Sigma_c]$ to prove our theorem.

We give an obvious lemma, which will be used in proving Theorem 1.1,

**Lemma 2.1** The second condition in (1.3) of Theorem 1.1, i.e.

$$\mu^3 \frac{f(x, u, \frac{p}{\mu})}{B(u, \frac{1}{\mu})} \text{ is concave in } (x, \mu) \in \mathbb{R}^n \times \mathbb{R}^+,$$ 

implies the quadratic form

$$\left[ \frac{f_{uuu} p_1 p_1}{B \mu^4} - 2 \frac{f_{uu} p_1 B_t}{B^2 \mu^3} - \frac{f B_{tt}}{B^2 \mu^4} + 2 \frac{f B_t^2}{B^3 \mu^4} - 4 \frac{f_{uu} p_t}{B \mu^3} + 4 \frac{f B_t}{B^2 \mu^3} + 6 \frac{f}{B \mu^2} \right] Y^2$$

$$+ \left[ 2 \frac{f_{sk} B_t}{B^2 \mu^2} + 6 \frac{f_{sk}}{B \mu} - 2 \frac{f_{uuu} p_t}{B \mu^2} \right] Y Z_k + \frac{f_{sksk}}{B} Z_i Z_k \leq 0, \quad \forall ((Z_i), Y) \in \mathbb{R}^n \times \mathbb{R}.$$ 

(2.4)

where $u_i = p_i/\mu$, $i = 1/\mu$.

**Proof of Theorem 1.1** Given a constant $c$, let $k$ be the minimum rank of the second fundamental forms on $\Sigma_c (k \in \{0, 1, \ldots, n - 1\})$. In the case $k = n - 1$ we are done. We will treat the case $1 \leq k \leq n - 2$ first, then indicate how to modify the argument in the case $k = 0$. Suppose the minimum rank $k$ arrive at point $z_0 \in \Sigma_c$, we consider the function $\phi = |Du|^{k+1} H_{k+1}[\Sigma_c]$, then $\phi(z_0) = 0$. Next we shall show that there exists a sufficiently small open neighborhood $O$ of $z_0$ such that $\phi(z) \equiv 0$ in $O$. If it is true then $\{ z \in \Sigma_c, \phi(z) = 0 \}$ is open. The set $\{ \phi = 0 \}$ is obviously closed in $\Sigma_c$, so we can conclude from connectivity that the rank of the level set second fundamental forms is constant of $k$.

First we notice that $B \neq 0$ and $\frac{A}{B} > 0$ by hypothesis. So we rewrite equation (1.2) as follows

$$\frac{A}{B}(u, |Du|) \left( \Delta u - \frac{u_{\alpha u} u_{\beta}}{|Du|^2} u_{\alpha \beta} \right) + \frac{u_{\alpha u} u_{\beta}}{|Du|^2} u_{\alpha \beta} = \frac{f(x, u, Du)}{B(u, |Du|)}$$

(2.5)
and denote the linear operator

\[ Fu = C^{\alpha\beta} u_{\alpha\beta} \]  

(2.6)

where \( C^{\alpha\beta} = \frac{A}{B} \delta_{\alpha\beta} + (1 - \frac{A}{B}) \frac{u_\alpha u_\beta}{|Du|^2} \). Clearly it is a uniformly elliptic operator. Now we need to prove the local estimates

\[ F \phi = C^{\alpha\beta} \phi_{\alpha\beta} \leq c_1 \phi + c_2 |\nabla \phi| \]  

(2.7)

Together with hypothesis \( \phi \geq 0 \) and \( \phi(z_0) = 0 \), we show \( \phi \equiv 0 \) in \( O \) by the strong maximum principle, from which the desired theorem will follow.

Following [3] here and below, we make the notations that functions \( h(y) \lesssim k(y) \) for \( y \in O \) provided there exist positive constants \( c_1 \) and \( c_2 \) such that

\[ (h - k)(y) \leq (c_1 \phi + c_2 |\nabla \phi|)(y). \]  

(2.8)

We also write \( h(y) \sim k(y) \) if \( h(y) \lesssim k(y) \) and \( k(y) \lesssim h(y) \). Next, we write \( h \lesssim k \) if the above inequality holds in \( O \), with the constant \( c_1 \), and \( c_2 \) independent of \( y \) in this neighborhood. Finally, \( h \sim k \) if \( h \lesssim k \) and \( k \lesssim h \). Hence, in this notation, we desire to make the local estimate \( F \phi \lesssim 0 \) in \( O \).

Since the rank at \( z_0 \) is \( k \) and the level set is convex, there must exists a constant \( C \in (0, 1] \) such that

\[ H_m[\Sigma_c] \gtrsim C, \quad 0 \leq m \leq k, \]  

(2.9)

\[ H_m[\Sigma_c] \sim 0, \quad k + 1 \leq m \leq n - 1 \]  

(2.10)

in sufficiently small neighborhood \( O \) of \( z_0 \). Now for each \( z \in O \) fixed, we first choose \( e_n \) such that \( |D u| = -u_n \). Then rotate coordinate \( e_1, \ldots, e_{n-1} \) such that the matrix \( W = (u_{i j}), i, j = 1, \ldots, n - 1 \) is diagonal and without loss of generality we may assume \( u_{11} \geq u_{22} \geq \cdots \geq u_{n-1n-1} \). Obviously we get for \( m = 0, \ldots, n - 1 \)

\[ H_m[\Sigma_c] = \frac{\partial S_m+1}{\partial u_{i j}} (D^2 u) u_i u_j \frac{1}{|D u|^m} = \frac{S_m(W)}{|D u|^m}, \]  

(2.11)

from which (2.9) and (2.10) implies \( u_{11} \geq u_{22} \geq \cdots \geq u_{kk} > C > 0 \), and \( u_{jj} \sim 0 \), \( j = k + 1, \ldots, n - 1 \). For convenience we denote \( G = \{1, \ldots, k\} \) and \( B = \{k + 1, \ldots, n - 1\} \) which means good terms and bad ones in indices separately. Without confusion we simply denote \( \{u_{11}, \ldots, u_{kk}\} = G \) and \( \{u_{k+1k+1}, \ldots, u_{n-1n-1}\} = B \).

In the following, all calculations are at the point \( z \) using the relation “\( \lesssim \)”, with the understanding that the constants in (2.8) are under control depending only on \( |u|_{C^4} \) and \( n \). Now we make the estimates for \( \phi \) and its derivatives in \( x_\alpha, x_\beta \).

\[ 0 \sim \phi = \frac{\partial S_{k+2}}{\partial u_{i j}} (D^2 u) u_i u_j = S_{k+1}(W) u_n^2 \sim S_k(G) S_1(B) u_n^2. \]  

(2.12)

From now on we denote the symmetric matrix \( \tilde{W} = (u_{i j}), i, j = 1, \ldots, n \) and \( Q = S_k(G) \) for simplicity. We also mention that the summing indices \( i, j \in \{1, \ldots, n\} \) if \( u_{i j} \in \tilde{W} \) and \( i, j \in \{1, \ldots, n - 1\} \) if \( u_{i j} \in W \).
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(2.12) yields that, for $1 \leq m \leq k$,

$$S_m(W) \sim S_m(G), \quad S_m(W|j) \sim \begin{cases} S_m(G|j), & \text{if } j \in G, \\ S_m(G), & \text{if } j \in B, \end{cases}$$

(2.13)

$$S_m(W|ij) \sim \begin{cases} S_m(G|ij), & \text{if } i, j \in G, \\ S_m(G|j), & \text{if } i \in B, j \in G, \\ S_m(G), & \text{if } i \neq j \in B, \end{cases}$$

(2.14)

Since $W$ is diagonal, it follows from [9],

$$\frac{\partial S_{\gamma}(W)}{\partial u_{ij}} = \begin{cases} S_{\gamma-1}(W|i), & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

(2.15)

and

$$\frac{\partial^2 S_{\gamma}(W)}{\partial u_{ij} \partial u_{rs}} = \begin{cases} S_{\gamma-2}(W|ir), & \text{if } i = j, r = s, i \neq r, \\ -S_{\gamma-2}(W|ir), & \text{if } i = s, j = r, i \neq j, \\ 0, & \text{otherwise} \end{cases}$$

(2.16)

for $1 \leq \gamma \leq n - 1$.

From (2.12) to (2.16), we have

$$0 \sim \phi_{\alpha} = \frac{\partial \phi}{\partial x_{\alpha}} = \frac{\partial^2 S_{k+2}}{\partial u_{ij} \partial u_{rs}}(\tilde{W})u_{ij}u_{rs\alpha} + 2 \frac{\partial S_{k+2}}{\partial u_{ij}}(\tilde{W})u_{i\alpha}u_{j}

= \frac{\partial S_{k+1}}{\partial u_{rs}}(W)u_{rs\alpha}u_{n}^2 + 2 \frac{\partial S_{k+2}}{\partial u_{in}}(\tilde{W})u_{i\alpha}u_{n}

= \frac{\partial S_{k+1}}{\partial u_{rs}}(W)u_{rs\alpha}u_{n}^2 - 2S_k(W|i)u_{in}u_{i\alpha}u_{n} + 2S_{k+1}(W)u_{n\alpha}u_{n}

\sim Q \sum_{j \in B} u_{j\alpha}u_{n}^2 - 2Q \sum_{j \in B} u_{j\alpha}u_{jn}u_{n}$$

which means

$$\sum_{j \in B} u_{j\alpha} \sim 0, \quad \forall \alpha \neq n, \quad \text{(2.17)}$$

$$\sum_{j \in B} u_{j\beta}u_{jn}u_{n} \sim 2 \sum_{j \in B} u_{jn}^2, \quad \text{(2.18)}$$

Then

$$\phi_{\alpha\beta} \sim \frac{\partial^2 S_{k+2}}{\partial u_{ij} \partial u_{rs}}(\tilde{W})u_{ij}u_{rs\alpha\beta} + \frac{\partial^3 S_{k+2}}{\partial u_{ij} \partial u_{rs} \partial u_{ml}}(\tilde{W})u_{ij}u_{rs\alpha\beta}u_{ml\beta}$$

$$+ 2 \frac{\partial^2 S_{k+2}}{\partial u_{ij} \partial u_{rs}}(\tilde{W})u_{i\beta}u_{jrs\alpha} + 2 \frac{\partial^2 S_{k+2}}{\partial u_{ij} \partial u_{rs}}(\tilde{W})u_{iau_{jrs\beta}}$$

$$+ 2 \frac{\partial S_{k+2}}{\partial u_{ij}}(\tilde{W})u_{i\alpha u_{j\beta}} + 2 \frac{\partial S_{k+2}}{\partial u_{ij}}(\tilde{W})u_{iau_{j\beta}}$$

(2.19)

We will divide $\phi_{\alpha\beta}$ into five terms to estimate $F \phi = C^{\alpha\beta} \phi_{\alpha\beta}$. Note $(C^{\alpha\beta})$ is symmetric and in fact diagonal, So
\[ I = C_{\alpha\beta} \frac{\partial^2 S_{k+2}}{\partial u_{ij} \partial u_{rs}} (\tilde{W})_{ui u j u rs\alpha\beta} \]
\[ = C_{\alpha\alpha} \frac{\partial S_{k+1}}{\partial u_{rs}} (W)_{u rs\alpha\alpha} \]
\[ \sim Q u_n^2 \sum_{j \in B} C_{\alpha\alpha} u_{\alpha j j}. \] (2.20)

\[ II = C_{\alpha\beta} \frac{\partial^3 S_{k+2}}{\partial u_{ij} \partial u_{rs} \partial u_{ml}} (\tilde{W})_{ui u j u rs u ml\beta} \]
\[ = C_{\alpha\alpha} \frac{\partial^2 S_{k+1}}{\partial u_{rs} \partial u_{ml}} (W)_{u rs\alpha\alpha} \]
\[ \sim 2 \sum_{i \in G, j \in B} S_{k-1}(G|i) C_{\alpha\alpha} u_{iia} u_{jj} u_{n}^2 - 2 \sum_{i \in G, j \in B} S_{k-1}(G|i) C_{\alpha\alpha} u_{i ja} u_{n}^2 \]
\[ + \sum_{i \neq j \in B \mid |B| \geq 2} S_{k-1}(G) C_{\alpha\alpha} u_{iia} u_{jj} u_{n}^2 - \sum_{i \neq j \in B \mid |B| \geq 2} S_{k-1}(G) C_{\alpha\alpha} u_{ija} u_{n}^2. \] (2.21)

\[ III = 2C_{\alpha\beta} \left[ \frac{\partial^2 S_{k+2}}{\partial u_{ij} \partial u_{rs}} (\tilde{W})_{ui u j u rs} + \frac{\partial^2 S_{k+2}}{\partial u_{ij} \partial u_{rs}} (\tilde{W})_{ui u j u rsf} \right] \]
\[ = 4C_{\alpha\alpha} \frac{\partial^2 S_{k+2}}{\partial u_{in} \partial u_{rs}} (\tilde{W})_{ui a u n u rs\alpha}, \] (2.22)

which divided again into four parts:

\[ III_1 = 4C_{\alpha\alpha} \frac{\partial^2 S_{k+2}}{\partial u_{in} \partial u_{ml}} (\tilde{W})_{ui a u n u ni\alpha} \]
\[ = -4C_{\alpha\alpha} S_k(W|i) u_{ia u n u ni\alpha} \]
\[ \sim -4Q \sum_{j \in B} C_{\alpha\alpha} u_{ja u jna u n}, \]
\[ \sim -4Q \sum_{j \in B} C_{\alpha\alpha} u_{jn u jnn u n}. \] (2.23)

\[ III_2 = 4C_{\alpha\alpha} \sum_{j \neq n} \frac{\partial^2 S_{k+2}}{\partial u_{in} \partial u_{ji}} (\tilde{W})_{ui a u n u jia} \]
\[ = 4C_{\alpha\alpha} S_{k-1}(W|ij) u_{jn u jia u jia} \]
\[ \sim 4 \sum_{i \in G, j \in B} S_{k-1}(G|i) u_{jn C_{\alpha\alpha} u_{i a u jia u n} + 4 \sum_{i \in G, j \in B} S_{k-1}(G|i) u_{jn C_{\alpha\alpha} u_{ja u jia u n} \]
\[ + 4 S_{k-1}(G) \sum_{i \neq j \in B \mid |B| \geq 2} C_{\alpha\alpha} u_{ia u jna u n} \]
\[ \sim 4 \sum_{i \in G, j \in B} S_{k-1}(G|i) u_{jn C_{\alpha\alpha} u_{ja u jia u n} + 8 \sum_{i \in G, j \in B} S_{k-1}(G|i) C_{\alpha\alpha} u_{jn u jnn u n} \]
\[ + 4 S_{k-1}(G) \sum_{i \neq j \in B \mid |B| \geq 2} C_{\alpha\alpha} u_{j n u jnn u n}. \] (2.24)
\[ III_3 = 4C^{\alpha\alpha} \sum_{i \neq n} \frac{\partial^2 S_{k+2}}{\partial u_{i\alpha} \partial u_{jj}} (\tilde{W}) u_{i\alpha} u_n u_{j\alpha} \]
\[ = -4C^{\alpha\alpha} S_{k-1}(W|j) u_{i\alpha} u_n u_{j\alpha} \]
\[ \sim -4 \sum_{i \in G} \sum_{j \in B} S_{k-1}(G|i) u_{n\alpha} C^{\alpha\alpha} u_{i\alpha} u_{j\alpha} u_n - 4 \sum_{i \in G} \sum_{j \in B} S_{k-1}(G|i) u_j C^{\alpha\alpha} u_{i\alpha} u_{j\alpha} u_n \]
\[ \sim -4 \sum_{i \in G} \sum_{j \in B} S_{k-1}(G|i) u_{i\alpha} u_j C^{\alpha\alpha} u_{i\alpha} u_{j\alpha} u_n - 4 \sum_{i \in G} \sum_{j \in B} S_{k-1}(G|i) u_j C^{\alpha\alpha} u_{i\alpha} u_{j\alpha} u_n \]
\[ \sim -4 \sum_{i \in G} \sum_{j \in B} \sum_{|B| \geq 2} C^{\alpha\alpha} u_{i\alpha} u_j u_{j\alpha} u_n \]
\[ III_4 = 4C^{\alpha\alpha} \frac{\partial^2 S_{k+2}}{\partial u_{n\alpha} \partial u_{jj}} (\tilde{W}) u_{n\alpha} u_{j\alpha} \]
\[ = 4C^{\alpha\alpha} S_k(W|j) u_{n\alpha} u_{j\alpha} \]
\[ \sim 4Q \sum_{j \in B} C^{\alpha\alpha} u_{n\alpha} u_{j\alpha} u_n. \]
some notations. For any function $F$

So we differentiate Eq. 2.5 in

Remember that the coefficient in (2.5) is

So we differentiate Eq. 2.5 in $x_j (j \in B)$ once to obtain

$$
C^{\alpha\beta} u_{\alpha\beta j} + [C^{\alpha\beta}_{u} u_{j} + C^{\alpha\beta}_{u l} u_{l j}] u_{\alpha\beta} = \frac{f_{x j}}{B} u_{j} + \left( \frac{f}{B} \right) u_{j} + \left( \frac{f}{B} \right) u_{l j}.
$$

(2.32)
Since $j \in B$ we directly get
\[ C^{\alpha \alpha} u_{\alpha j} + C^{\alpha \alpha}_{u_n} u_{nj} u_{\alpha \alpha} \sim \frac{f x_j}{B} + \left( \frac{f}{B} \right) u_{nj}. \] (2.33)

Differentiate Eq. 2.32 again in $x_j$ one can see
\[ C^{\alpha \alpha} u_{\alpha j} + 2 C^{\alpha \alpha}_{u_n} u_{nj} u_{\alpha j} + \left( C^{\alpha \alpha}_{u_n} u_{nj} + C^{\alpha \alpha}_{u_n u_n} u_{nj}^2 \right) u_{\alpha \alpha} \]
\[ \sim \frac{f x_{x_j}}{B} + 2 \left( \frac{f}{B} \right) u_{nj} + \left( \frac{f}{B} \right) u_{nj}^2 + \left( \frac{f}{B} \right) u_{njj}. \] (2.34)

Let’s denote $t = |Du|$ for convenience in computations below and recall $t = -u_n$:
\[ C^{\alpha \alpha} = \frac{A}{B} (u, t) + \left[ 1 - \frac{A}{B} (u, t) \right] \delta_{an}, \] (2.35)

and $t u_n = -1$, $(u_{\alpha}/t)_{u_n} = 0$, $(u_{\alpha}/t)_{u_n u_n} = 0$, so
\[ C^{\alpha \alpha}_{u_n} = (1 - \delta_{an}) \left( \frac{A}{B} \right)_t, \quad C^{\alpha \alpha}_{u_n u_n} = (1 - \delta_{an}) \left( \frac{A}{B} \right)_t. \] (2.36)

\[ \left( \frac{f}{B} \right)_{u_n} = \frac{f u_n}{B} + f B t \frac{B^2}{B^2}, \] (2.37)

\[ \left( \frac{f}{B} \right)_{u_n u_n} = \frac{f u_n u_n}{B} + 2 \frac{f u_n B t}{B^2} - \frac{B f B t^2}{B^2} + 2 \frac{B^2}{B^3}, \] (2.38)

\[ \left( \frac{f}{B} \right)_{u_n x_j} = \frac{f u_n x_j}{B} + f x_j B t \frac{B^2}{B^2}. \] (2.39)

Now we can calculate $I - V$ more carefully by using (2.17) to (2.39) and all the facts above. Firstly from (2.34) to (2.39) we have
\[
I \sim Q u_n^2 \sum_{j \in B} C^{\alpha \alpha}_{u_n u_n} u_{\alpha j} u_{jj}
\[
\sim -Q u_n^2 \sum_{j \in B} \left[ -2 \left( \frac{A}{B} \right)_t \sum_{i \in G} u_{ij} u_{jn} + 2 \left( \frac{A}{B} \right)_t \sum_{i \in G} u_{ij} u_{jn}^2 + \left( \frac{A}{B} \right)_t \sum_{i \in G} u_{ij}^2 u_{jn} \right]
\[
+ Q u_n^2 \sum_{j \in B} \left[ \frac{f u_n u_n}{B} + 2 \frac{f u_n B t}{B^2} - \frac{B f B t^2}{B^2} + 2 \frac{B^2}{B^3} \right] u_{jn}^2
\]
\[
+ 2 \left( \frac{f u_n}{B} + f B t \frac{B^2}{B^2} \right) u_{jn}^2 + 2 \left( \frac{f u_n x_j}{B} + f x_j B t \frac{B^2}{B^2} \right) u_{jn} + \frac{f x_j x_j}{B}. \] (2.40)

Then the following by (2.17), (2.18), (2.35)
\[
II \sim 2 \sum_{i \in G} S_{k-1}(G|i) C^{\alpha \alpha}_{u_{ij} u_{ija}} u_{j ij} u_{nj}^2 - 2 \sum_{i \in G} S_{k-1}(G|i) C^{\alpha \alpha}_{u_{ij} u_{ija}} u_{nj}^2
\]
\[
+ \sum_{i \neq j \in B \mid |B| \geq 2} S_{k-1}(G) C^{\alpha \alpha}_{u_{ija} u_{ija}} u_{nj}^2 - \sum_{i \neq j \in B \mid |B| \geq 2} S_{k-1}(G) C^{\alpha \alpha}_{u_{ija} u_{ija}} u_{nj}^2
\]
\[
\sim 4 \sum_{i \in G} S_{k-1}(G|i) C^{nn}_{u_{ij} u_{ija} u_{ij} u_{ija}} u_{nj}^2 - 2 \sum_{i \in G} S_{k-1}(G|i) C^{\alpha \alpha}_{u_{ija} u_{ija}} u_{nj}^2
\]
\[
\sum_{i \in B \mid |B| \geq 2} S_{k-1}(G)C^{\alpha\alpha} u_{ii}\left(\sum_{j \in B \mid |B| \geq 2} u_{jj} - u_{ii}\right) u_n^2
\]

\[-S_{k-1}(G)u_n^2 \sum_{i \neq j \in B \mid |B| \geq 2} C^{\alpha\alpha} u_{ij}\]

\[\sim 4 \sum_{i \in G \mid j \in B} S_{k-1}(G|i)u_{ii}u_{jn}u_n^2 - 2 \sum_{i \in G \mid j \in B} S_{k-1}(G|i)C^{\alpha\alpha} u_{ij}u_n^2 \]

\[+4S_{k-1}(G) \left( \sum_{j \in B \mid |B| \geq 2} u_{jn}^2 \right) - S_{k-1}(G) \sum_{j \in B \mid |B| \geq 2} C^{\alpha\alpha} u_{ji}u_n^2 \]

\[-S_{k-1}(G)u_n^2 \sum_{i \neq j \in B \mid |B| \geq 2} C^{\alpha\alpha} u_{ij}. \quad (2.41)\]

Together with (2.33)

\[III_1 \sim -4Q \sum_{j \in B} C^{nn} u_{jn}u_{jjn}u_n\]

\[\sim 4Q \sum_{j \in B} \left[ \frac{A}{B} \sum_{i \in G} u_{ii} - \left( \frac{A}{B} \right) \sum_{i \in G} u_{ii}u_{jn} \right] u_{jn}u_n\]

\[\sim 4Q \sum_{j \in B} \left[ \frac{f_{xj}}{B} + \left( \frac{f_{nn}}{B} + \frac{f_{B^2}}{B^2} \right) u_{jn} \right] u_{jn}u_n \quad (2.42)\]

\[III_2 \sim 4 \frac{A}{B} \sum_{i \in G \mid j \in B} S_{k-1}(G|i)u_{ii}u_{ij}u_{jn}u_n + 8 \sum_{i \in G \mid j \in B} S_{k-1}(G|i)u_{in}u_{jn}u_{ijn}u_n \]

\[+4S_{k-1}(G) \sum_{i \neq j \in B \mid |B| \geq 2} u_{in}u_{jn}u_{ijn}u_n. \quad (2.43)\]

\[III_3 \sim -8 \sum_{i \in G \mid j \in B} S_{k-1}(G|i)u_{jn}^2u_{in}^2 - 4 \sum_{i \in G \mid j \in B} S_{k-1}(G|i)u_{jn}u_{in}u_{jn}u_n\]

\[-4S_{k-1}(G) \sum_{i \in B \mid |B| \geq 2} u_{jn}^2 \left( \sum_{j \in B \mid |B| \geq 2} u_{jj} - u_{ii} \right) u_n\]

\[\sim -8 \sum_{i \in G \mid j \in B} S_{k-1}(G|i)u_{jn}^2u_{jn}^2 - 4 \sum_{i \in G \mid j \in B} S_{k-1}(G|i)u_{jn}^2u_{in}u_{jn}u_n\]

\[-8S_{k-1}(G) \left( \sum_{j \in B \mid |B| \geq 2} u_{jn}^2 \right)^2 + 4S_{k-1}(G) \sum_{j \in B \mid |B| \geq 2} u_{jn}u_{jn}u_n. \quad (2.44)\]
Combine $III_4$ with $V = V_1 + V_2 + V_3$ we simply get

$$III_4 + V \sim 6Qc_{nn}u_{nn} \sum_{j \in B} u_{jn}^2 - 2 \sum_{i \in G, j \in B} S_{k-1}(G|i)C^{ii}u_{ii}^2u_{jn}^2$$

\[ \sim 6Q \left( \frac{f}{B} - \frac{A}{B} \sum_{i \in G} u_{ii} \right) \sum_{j \in B} u_{jn}^2 - 2 \frac{A}{B} \sum_{i \in G, j \in B} S_{k-1}(G|i)u_{ii}^2u_{jn}^2. \tag{2.45} \]

where we used Eq. 2.5. And also by (2.33)

$$IV \sim 2Q \sum_{j \in B} \left[ - \left( \frac{A}{B} \right) \sum_{i \in G} u_{ii} u_{jn} - \left( \frac{f_{un}}{B} + \frac{f_{Bt}}{B^2} \right) u_{jn} - \frac{f_{xj}}{B} \right] u_{jn} u_n. \tag{2.46}$$

So far we have prepared everything to estimate $F \phi$. Let us combine $I - V$ in (2.40) to (2.46) with understanding $u_n = -t$, $Q = S_{k-1}(G|i)u_{ii}$, $\forall i \in G$, so:

$$F \phi \sim \tilde{I} + \tilde{II} + \tilde{III}, \tag{2.47}$$

where

$$\tilde{I} \sim 2Q \sum_{j \in B} \left[ - \left( \frac{A}{B} \right) \sum_{i \in G} u_{ii} u_{jn} - \left( \frac{f_{un}}{B} + \frac{f_{Bt}}{B^2} \right) u_{jn} - \frac{f_{xj}}{B} \right] u_{jn}$$

\[ + \left[ 2 \left( \frac{f_{nxj}}{B} + \frac{f_{Bj}}{B^2} \right) u_{jn} - \frac{f_{xj}}{B} \right] u_{jn} u_n + \frac{f_{xj}u_{jn}^2}{B^2}, \tag{2.48} \]

which comes from parts of terms $I, III_1, IV$ and $III_4 + V$. The left parts together with the first term in $III_2$ (2.43) and the second term in $II(2.41)$ for $\alpha = i$ give

$$\tilde{II} \sim \sum_{i \in G, j \in B} S_{k-1}(G|i) \left\{ \left[ \left( \frac{A}{B} \right) t \right] \sum_{i} u_{ii}^2u_{jn}^2 - 4 \left( \frac{A}{B} \right) t + 8 \frac{A}{B} \right\}$$

\[ + \left[ 2 \left( \frac{A}{B} \right) t - 8 \frac{A}{B} \right] u_{ij} u_{jn} u_{ij} u_{nn} + \frac{2A^2}{B} u_{ij}^2 u_{nn}^2. \tag{2.49} \]

and the left terms in $II, III_2$ together with $III_3$ gives

$$\tilde{III} \sim -2 \sum_{i \in G, j \in B, \alpha \neq i} S_{k-1}(G|i)C^{\alpha} u_{ij} u_{nn}^2 - 4S_{k-1}(G)( \sum_{j \in B} u_{jn}^2 )^2$$

\[ - S_{k-1}(G) \sum_{j \in B, |B| \geq 2} \sum_{\alpha \in B} C^{\alpha} u_{ij}^2 u_{nn}^2 - S_{k-1}(G) u_{nn}^2 \sum_{j \in B, |B| \geq 2} C^{\alpha} u_{ij}^2 \]

\[ + 8 \sum_{i \in G, j \in B} S_{k-1}(G|i) u_{in} u_{jn} u_{ij} u_{nn} + 4 S_{k-1}(G) \sum_{i \neq j \in B, |B| \geq 2} u_{ij} u_{jn} u_{ij} u_{nn} \]

\[ - 8 \sum_{i \in G, j \in B} S_{k-1}(G|i) u_{in} u_{jn}^2 + 4 S_{k-1}(G) \sum_{j \in B, |B| \geq 2} u_{jn}^2 u_{ij} u_{nn}. \tag{2.50} \]

Firstly by the second hypothesis in (1.3), we set $Z_i = \delta_{ij} u_{nn}, Y = u_{jn}/u_n$ in Lemma 2.1. Note at point $z, |Du| = -u_n, s o \mu = -u_n^{-1}, p = (0, \ldots, -1)$ in this lemma. Then we easily know $\tilde{I} \lesssim 0$ since $Q \geq C > 0$. \[

\frac{1}{2}
\]
In order to get $\tilde{I} \lesssim 0$, of course its coefficients must satisfy
\[
\left( \frac{A}{B} \right)_{tt} t^2 - 4 \left( \frac{A}{B} \right)_t t + 8 \frac{A}{B} \geq 0,
\]  
(2.51)
and
\[
\left[ \left( \frac{A}{B} \right)_{tt} t^2 - 4 \left( \frac{A}{B} \right)_t t + 8 \frac{A}{B} \right] \cdot 2 \frac{A}{B} - \left[ \left( \frac{A}{B} \right)_t - 4 \frac{A}{B} \right] ^2 \geq 0.
\]  
(2.52)
In fact
\[
\left( \frac{A}{B} \right)_{tt} t^2 - 4 \left( \frac{A}{B} \right)_t t + 8 \frac{A}{B} = \left[ \left( \frac{A}{B} \right)_{tt} - 1 \right] - \frac{1}{2} \left( \frac{A}{B} \right)_t t^2 - \left( \frac{A}{B} \right)_t - 4 \frac{A}{B} \right] \cdot 2 \frac{A}{B} - \left[ \left( \frac{A}{B} \right)_t - 4 \frac{A}{B} \right] ^2 \right\}
\]  
(2.53)
and
\[
\left[ \left( \frac{A}{B} \right)_{tt} t^2 - 4 \left( \frac{A}{B} \right)_t t + 8 \frac{A}{B} \right] \cdot 2 \frac{A}{B} - \left[ \left( \frac{A}{B} \right)_t - 4 \frac{A}{B} \right] ^2 \geq 0.
\]  
(2.54)
So inequalities (2.51) and (2.52) hold if
\[
2 \frac{A}{B} \left( \frac{A}{B} \right)_{tt} - \left( \frac{A}{B} \right)_t ^2 \geq 0,
\]  
(2.55)
which comes from the first condition in (1.3).

Using Cauchy inequality to deal with three “positive” terms in $\tilde{III}$, then
\[
\tilde{III} \lesssim -2 \frac{A}{B} \sum_{i \in G, j \in B} S_{k-1}(G|G) u_{ij} u_n^2 - \frac{A}{B} \sum_{j \in B, |B| \geq 2} u_{jj}^2
- \frac{A}{B} S_{k-1}(G) u_n^2 \sum_{i \neq j, |B| \geq 2} C_{ij} u_{ij}^2
\]  
(2.56)
Finally we have shown $F \phi \lesssim 0$ by (2.47), i.e. our theorem holds. At the beginning of our proof, we consider $k = 1, \ldots, n - 2$. For $k = 0$ the estimate only have $\tilde{I}$, the theorem holds obviously.

Remark 2.2 In the proof of constant rank theorem, we use the expression (2.2) for the elementary symmetric functions of the principal curvatures of the level surface which is different to Korevaar [12]. We will consider constant rank theorem for Hessian type equations later.

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