Convexity estimates for level sets of quasiconcave solutions to fully nonlinear elliptic equations

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Abstract. We establish a global geometric lower bound for the second fundamental form of the level surfaces of solutions to $F(D^2u, Du, u, x) = 0$ in convex ring domains, in terms of boundary geometry and the structure of the elliptic operator $F$. We also prove a microscopic constant rank theorem, under a general structural condition introduced by Bianchini–Longinetti–Salani in 2009.

1. Introduction

Solutions of boundary value problems for elliptic equations often inherit important geometric properties of the domains with the influence of the structures of the corresponding equations. One of these geometric features is the quasiconcavity. A function $u$ is called quasiconcave if its level sets $\{x : u(x) \geq c\}$ are convex. By the work of Gabriel [8], the Green function of a convex domain is quasiconcave. The same is also true for $p$-harmonic functions in convex ring domains with homogeneous boundary conditions following Lewis [12]. Another example is the quasiconcavity of solutions to the free boundary problem arising in plasma physics in convex domains in the work of Caffarelli–Spruck [6]. The quasiconcavity of solutions to nonlinear equations has been studied extensively in the literature, we refer to [2], [3], [5]–[14] and references therein. The techniques of quasiconcave envelopes have been refined by Colesanti–Salani [6], and more recently by Bianchini–Longinetti–Salani [3] to prove quasiconcavity of solutions to general degenerate elliptic fully nonlinear equations in the form

$$F(D^2u, Du, u, x) = 0,$$

in the convex ring domain $\Omega = \Omega_0 \setminus \Omega_1$ (i.e. $\Omega_0 \subset \subset \Omega_1$ are convex) with the Dirichlet boundary condition

$$u|_{\partial \Omega_0} = 0 \quad \text{and} \quad u|_{\partial \Omega_1} = 1.$$

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The main focus of this paper is on the quantified properties of the quasiconcave solutions of equations of form (1.1). More specifically, we establish a global a priori estimate on the geometric lower bound of the principal curvatures of the level surfaces of these quasiconcave solutions, in terms of boundary geometry and the structure of operator $F$. In addition to the geometric interest, this type of estimates may be used via homotopic deformation to obtain the existence of quasiconcave solutions of the corresponding equations. We achieve this macroscopic geometric estimate through a microscopic constant rank theorem for the smallest principal curvatures of the level surfaces for quasiconcave solutions. A simple microscopic convexity principle for level surfaces of solutions of equations of the form (1.1) is obtained in Theorem 1.3, under a general structural condition introduced in [3] to cover a larger class of nonlinear equations. A more refined version for the smallest principal curvatures of the level surfaces for quasiconcave solutions is proved in the last section of the paper. The main result, Theorem 1.4, is a consequence of this type of microscopic principle.

Let us introduce some notation. Denote $\mathcal{S}_n$ the space of real symmetric $n \times n$ matrices and let $Y \subset \mathcal{S}_n$ be an open set.

**Definition 1.1.** For all $\theta \in \mathbb{S}^{n-1}$, denote $\theta^\perp$ the linear subspace in $\mathbb{R}^n$ which is orthogonal to $\theta$. Define $\mathcal{S}_n^-(\theta)$ to be the class of $n \times n$ symmetric real matrices which are negative definite on $\theta^\perp$. Denote $\mathcal{S}_n^{0-} (\theta)$ the subclass of $\mathcal{S}_n^-(\theta)$ of matrices that have $\theta$ as eigenvector with corresponding null eigenvalue. For any $b \in \mathbb{R}^n$ with $t = \langle b, \theta \rangle > 0$, define

$$\mathcal{B}_n^-(\theta) = \left\{ B \in \mathcal{S}_{n+1} : B = \begin{pmatrix} \tilde{B} & b^T \\ b & \chi \end{pmatrix} \right\},$$

with $\tilde{B} \in \mathcal{S}_n^{0-} (\theta) \cap Y, \chi \in \mathbb{R}$.

Denote by $J = (I_n \mid 0)$ the $n \times (n + 1)$ matrix, where $I_n$ is the $n \times n$ identity matrix and $0$ is the null vector in $\mathbb{R}^n$. Suppose $F = F(r,p,u,x)$ is a $C^2$ function in $Y \times \mathbb{R}^n \times \mathbb{R} \times \Omega$ for all $(\theta,u) \in \mathbb{S}^{n-1} \times \mathbb{R}$ fixed, set

$$\Gamma_F = \{ (B,x) \in \mathcal{B}_n^-(\theta) \times \Omega : F(t^{-1}JB^{-1}JT, t^{-1}\theta, u, x) \geq 0 \}.$$  

The following was proved in [3].

**Theorem 1.2** (Bianchini–Longinetti–Salani). Suppose $F$ is a proper, continuous, degenerate elliptic operator which satisfies a viscosity comparison principle. Assume that for each $(\theta,u)$ fixed, the super-level set $\Gamma_F$ defined in (1.4) is convex. If $u \in C^2(\Omega) \cap C(\overline{\Omega})$ with $|\nabla u| > 0$ is an admissible classical solution of equation (1.1) satisfying the Dirichlet boundary value (1.2) in the convex ring domain $\Omega$, then the level set $\{ x \in \Omega : u(x) \geq c \} \cup \Omega_1$ is convex for each constant $0 \leq c \leq 1$.

The class of operators $F$ satisfying the conditions in Theorem 1.2 includes Laplace operators, $p$-Laplace operators, the Pucci operator, and the mean curvature type equations of the form

$$\sum_{i,j=1}^n a_{ij}(\nabla u, u, x)u_{ij} = f(\nabla u, u, x).$$
A similar result was also proved by Bianchini–Longinetti–Salani in [3] under the assumption that for all \((\theta, u)\) fixed,

\[
\Xi_F = \{(A, t, x) \in \gamma \times (0, +\infty) \times \Omega : F(t^{-3}A, t^{-1}u, x) \geq 0\}
\]

is locally convex. With this structural condition on \(F\), a constant rank theorem was obtained in [2]. The convexity structural condition on \(F\) in Theorem 1.2 is weaker than the convexity structural condition on \(\Xi_F\). In particular, the mean curvature operator (1.5) does not satisfy condition (1.6). Detailed discussion of these conditions as well as examples will be given in Section 2.

To establish a strict convexity estimate on the second fundamental forms of the level surfaces of solutions in Theorem 1.2, we need two assumptions:

(1.7) Ellipticity: \((F^{\gamma}) := \left( \frac{\partial F}{\partial \gamma} (\nabla^2 u(x), \nabla u(x), u(x), x) \right) > 0, \quad \forall x \in \Omega.\)

(1.8) Structural condition: The set \(\Gamma_F\) is locally convex for all \((\theta, u)\) fixed.

Throughout the paper, we assume

\[
|Du(x)| \geq d_0 > 0, \quad \forall x \in \Omega,
\]

to ensure that the level-surface \(\{x \in \Omega : u(x) = c\}\) is smooth for each \(c\).

The first result of this paper is a microscopic constant rank theorem.

**Theorem 1.3.** Suppose \(u \in C^{3,1}(\Omega)\) is a solution of (1.1) and \((D^2u(x), Du(x), u(x)) \in \gamma \times \mathbb{R}^n \times (-\gamma_0 + \delta_0, \gamma_0 + \delta_0)\) for some \(\delta_0 \in \mathbb{R}\) at \(x \in \Omega\). Suppose that \(F\) satisfies conditions (1.7)–(1.8) and the level set \(\{x \in \Omega : u(x) \geq c\} \cup \Omega_1\) of \(u\) is connected and locally convex for all \(c \in (-\gamma_0 + \delta_0, \gamma_0 + \delta_0)\) for some \(\gamma_0 > 0\). Then the second fundamental form of the level surface

\[
\Sigma^c = \{x \in \Omega : u(x) = c\}
\]

has the same constant rank for all \(c \in (-\gamma_0 + \delta_0, \gamma_0 + \delta_0)\).

We now switch our attention to global geometric bounds of the second fundamental forms of level surfaces of \(u\). For a function \(u\) defined in domain \(\Omega\), denote by

\[
\Sigma^c = \{x \in \bar{\Omega} : u(x) = c\}
\]

the level surface. For any \(x \in \Sigma^c\), denote by \(\kappa_s(x)\) the smallest principal curvature of the level surface \(\Sigma^c\) at \(x\). For each \(c \in \mathbb{R}\), if \(\Sigma^c \neq \emptyset\), set

\[
\kappa^c = \inf_{x \in \Sigma^c} \kappa_s(x).
\]
We will strengthen (1.7) to

\[
(1.10) \quad \text{Uniform ellipticity: } \exists \lambda > 0, \quad \left( \frac{\partial F}{\partial \gamma} (D^2 u(x), Du(x), u(x), x) \right) \geq \lambda \langle \delta \gamma \rangle, \\
\forall x \in \Omega.
\]

Set

\[
(1.11) \quad \omega = \max \limits_{x, \beta, \eta, \gamma} \sup \left\{ \left[ \frac{\partial^2 F(D^2 u(x), Du(x), u(x), x)}{\partial r \partial \gamma} \right], \left[ \frac{\partial F(D^2 u(x), Du(x), u(x), x)}{\partial r \partial \gamma} \right] \cdot |Du(x)| \right\}.
\]

**Theorem 1.4.** Suppose \( u \) is a classical solution of equation (1.1) with the Dirichlet boundary value (1.2) in the convex ring domain \( \Omega \). Suppose \( F \) satisfies conditions (1.8)–(1.10) at \( (D^2 u, Du, u, x) \in Y \times \mathbb{R}^n \times [0, 1] \times \Omega \). Then

\[
(1.12) \quad \kappa^c \geq \min \left\{ \kappa_0 e^{-Ac}, \kappa_1 e^{A(c-1)}, \frac{2e^{A(c-1)}}{100\omega} \right\}, \quad \forall c \in [0, 1],
\]

for some universal constant \( A \geq 0 \) depending only on \( \|F\|_{\mathcal{C}^2}, n, \lambda, d_0, \|u\|_{\mathcal{C}^3} \).

It should be pointed out that the convexity estimates carried out in this paper are very sensitive to the structure of the corresponding equation. For equations of the form (1.5) with the Dirichlet boundary condition (1.2), the behavior of \( f \) is crucial. For instance, in the case of the Laplace equation

\[
(1.13) \quad \Delta u = f(u),
\]

Theorems 1.3 and 1.4 are true when \( f \geq 0 \). In general, Theorem 1.3 does not hold if \( f(u) < 0 \) in equation (1.13), even for \( f \equiv -1 \).

The rest of the paper is organized as follows. In Section 2, we discuss the structural conditions and prove two key lemmas (Lemmas 2.5 and 2.6). An auxiliary curvature test function is analyzed in Section 3. The proof of Theorems 1.3 and 1.4 is given in the last section, by establishing a strong maximum principle for the test function considered in Section 3.

### 2. Structural conditions

We recall some notation and results from [3].

**Definition 2.1.** For all \( \theta \in \mathbb{S}^{n-1} \), denote by \( \mathcal{A}^- (\mathcal{Y}) \) the following open set in \( \mathcal{S}_{n+1}^- \):

\[
(2.1) \quad \mathcal{A}^- (\mathcal{Y}) = \left\{ A \in \mathcal{S}_{n+1}^- : A = \begin{pmatrix} \Lambda & \mu \theta \\ \mu \theta^T & 0 \end{pmatrix} \text{ with } \Lambda \in \mathcal{S}_{n}^-(\theta) \cap \mathcal{Y}, \mu > 0 \right\},
\]
Properties of $\mathcal{A}_0$, $\mathcal{B}_0$ and their relationship have been studied in [3]. We list some of them which will be used in this paper. We have $\det A \neq 0$ if $A \in \mathcal{A}_0$, and

\begin{equation}
(2.2) \quad \mathcal{B}_0^-(Y) = \{ A^{-1} : A \in \mathcal{A}_0(Y) \}.
\end{equation}

If $B = A^{-1} \in \mathcal{B}_0^-(Y)$, then

\begin{equation}
(2.3) \quad \bar{A} = JB^{-1}J^T \quad \text{and} \quad \mu = \frac{1}{t}.
\end{equation}

Set

\begin{equation}
(2.4) \quad Q = t^2 JB^{-1}J^T,
\end{equation}

where $B \in \mathcal{B}_0^-(Y)$ and $t$ are defined as in Definition 1.1. By symmetry of $B$, we have

\begin{equation}
I_e^0 = F_{abc}q_{ab}B_{cd}q_{cd}X_{cd}X_{ef} \leq 0,
\end{equation}

for any nonnegative definite $n \times n$ matrix $(F^{ab})$ and any $(n+1) \times (n+1)$ symmetric matrix $(X_{cd})$.

**Proof.** The concavity of $Q$ has been proved in [3]. For any nonnegative definite $n \times n$ matrix $(F^{ab})$, there exist $\zeta_1, \ldots, \zeta_n \in \mathbb{R}^n$, such that

\begin{equation}
(F^{ab}) = \zeta_1^{T}v_1^{T} + \cdots + \zeta_n^{T}v_n^{T}.
\end{equation}

Therefore, $I \leq 0$ follows directly from the concavity of $Q$. \square

For the function $F(r, p, u, x)$, write $F^{ab} = \partial F / \partial r_{ab}$, $F^{pi} = \partial F / \partial p_i$ as derivatives of $F$ with respect to the corresponding arguments. For the level set $\Gamma_F$ defined in (1.4), denote the...
Condition (1.8) is equivalent to the fact

\[ V \nabla_{(B,x)}^2 F V^T \leq 0, \quad \forall V = (X_{cd}, (Z_k)) \in \mathcal{T}_F. \]

A straight computation yields,

\begin{align*}
\mathcal{V}_B F &= \left( \frac{F_{\beta \gamma}}{t^6} \left( \frac{\partial Q_{\beta \gamma}}{\partial B_{cd}} X_{cd} - 3 Q_{\beta \gamma} X_{n+1} \theta_t \right) - \frac{F_{\beta \gamma}}{t^2} \delta_{n+1} \delta_{n} \theta_t \theta_t \right), \\
\mathcal{V}_s F &= \left( F_{x_1}, \ldots, F_{x_6} \right),
\end{align*}

and

\[ V \nabla_{(B,x)}^2 F V^T = \frac{F_{\beta \gamma} \gamma_7}{t^6} \left( \frac{\partial Q_{\beta \gamma}}{\partial B_{cd}} X_{cd} - 3 Q_{\beta \gamma} X_{n+1} \theta_t \right) \left( \frac{\partial Q_{\beta \gamma}}{\partial B_{cf}} X_{cf} - 3 Q_{\beta \gamma} X_{n+1} \theta_t \right) \]

\[ - 2 \frac{F_{\beta \gamma} p_i}{t^6} \theta_t \left( \frac{\partial Q_{\beta \gamma}}{\partial B_{cd}} X_{cd} - 3 Q_{\beta \gamma} X_{n+1} \theta_t \right) X_{n+1} \theta_s \]

\[ + \frac{F_{\beta \gamma}}{t^3} \frac{\partial^2 Q_{\beta \gamma}}{\partial B_{cd} \partial B_{cf}} X_{cd} X_{cf} - 6 \frac{F_{\beta \gamma} Q_{\beta \gamma}}{t^5} X_{n+1} X_{n+1} \theta_t \theta_s \]

\[ - 6 \frac{F_{\beta \gamma}}{t^5} \left( \frac{\partial Q_{\beta \gamma}}{\partial B_{cd}} X_{cd} - 3 Q_{\beta \gamma} X_{n+1} \theta_t \right) X_{n+1} \theta_s \]

\[ + 2 \frac{F_{\beta \gamma} p_i}{t^3} X_{n+1} X_{n+1} \theta_t \theta_t + \frac{F_{\beta \gamma} p \theta_t \theta_t p}{t^3} X_{n+1} X_{n+1} \theta_t \theta_t \]

\[ + F_{x_1, x_6} Z_k Z_l + 2 \frac{F_{\beta \gamma} x_k}{t^6} \left( \frac{\partial Q_{\beta \gamma}}{\partial B_{cd}} X_{cd} - 3 Q_{\beta \gamma} X_{n+1} \theta_t \right) Z_k \]

\[ - 2 \frac{F_{\beta \gamma} x_1 \theta_l}{t^5} X_{n+1} \theta_s Z_k. \]

This expression suggests us to set

\[ \tilde{X}_{\beta \gamma} = t^{-4} \left( \sum_{c,d} \frac{\partial Q_{\beta \gamma}}{\partial B_{cd}} X_{cd} - 3 Q_{\beta \gamma} \sum_l X_{n+1} \theta_l \right), \quad \tilde{Y} = -t^{-2} \sum_s X_{n+1} \theta_s. \]

For

\[ V = ((X_{cd}), (Z_k)), \quad H(V, V) = V \nabla_{(B,x)}^2 F V^T, \]
$H(V, V)$ can be written as

$$
(2.10) \quad H(V, V) = F^{\beta;\gamma\eta} \tilde{X}_{\beta\eta} \tilde{X}_{\gamma\eta} + 2F^{\beta;\gamma\eta} 0_{\eta} \tilde{X}_{\beta\eta} \tilde{Y} + 2F^{\beta;\gamma\eta} \tilde{X}_{\gamma\eta} \tilde{Z}_{\eta} + F^{\beta;\gamma\eta} \tilde{Y} \tilde{Z}_{\eta} + 2tF^{\beta;\gamma\eta} 0_{\eta} \tilde{Y}^{2}
$$

$$
+ 6tF^{\beta;\gamma\eta} \tilde{X}_{\beta\eta} \tilde{Y} - 6t^{-1} F^{\beta;\gamma\eta} Q_{\beta\eta} \tilde{Y}^{2} + \frac{I}{t^{5}},
$$

where Einstein’s summation convention is used and $I$ is defined in (2.5). At this point, we have proved

**Lemma 2.3.** Condition (1.8) is equivalent to

$$
H(V, V) \leq 0 \quad \text{for all } V = ((X_{cd}), (Z_{\ell})) \in T \Gamma_{F},
$$

where $F^{\beta;\gamma\eta}$, $F^{\beta;\gamma\eta}$, etc. in (2.10) are evaluated at $(t^{-3}Q, t^{-1}0, u, x)$.

We may now compare conditions (1.8) and (1.6), these are the two structural conditions introduced in [3] as (3.10) and (1.2). As already discussed by Bianchini–Longinetti–Salani in [3], a variation of these two conditions can be compared ([3], Theorem 3.12). In fact, the following is true.

**Corollary 2.4.** The condition (1.6) that $X_{\beta}$ for each $\theta$, $u$ is locally convex implies condition (1.8).

**Proof.** Lemma 4.1 in [2] states that condition (1.6) implies $H(V, V) \leq t^{-3}I$ where $H(V, V)$ is defined in (2.10) and $I$ is defined in (2.5), respectively. The corollary follows directly from Lemma 2.2 and Lemma 2.3.

The quantity $I$ defined in (2.5) is a crucial term. We wish to compute this term explicitly, so it can be used in the proof of the main theorems in the last section. For our purpose, we set $\theta = (0, \ldots, 0, 1)$. In this case, $A$ and $B$ can be written as (see [3])

$$
(2.11) \quad A = \begin{pmatrix}
\times & 0 \\
\vdots & \vdots \\
\times & 0 \\
0 & \cdots & 0 & \cdots & 0 & \mu & 0
\end{pmatrix},
$$

$$
(2.12) \quad B = \begin{pmatrix}
0 & \times \\
\vdots & \vdots \\
0 & \times \\
0 & \cdots & 0 & \cdots & 0 & t \\
\times & \cdots & \times & t & \times
\end{pmatrix},
$$

where the $(n - 1) \times (n - 1)$ matrix $(a_{\beta})$ is negative definite and can be assumed diagonal, $(a^{\beta})$ is the inverse matrix of $(a_{\beta})$, $t = B_{n+1,n} = \frac{1}{\mu} > 0$. The values at the positions denoted by $\times$ are not important in the calculations.
Note that \( B_{ln} = B_{nl} = 0 \) for all \( l \leq n \). We may as well set
\[
X_{ln} = X_{nl} = 0, \quad \forall l \leq n.
\]
Denote
\[
B^{\alpha\beta} = (B^{-1})_{\alpha\beta} = A_{\alpha\beta}, \quad T := \{1, \ldots, n-1\}.
\]
We compute
\[
(2.13) \quad F^{\alpha\beta} \frac{\partial^2 Q_{\alpha\beta}}{\partial B_{cd} \partial B_{ef}} X_{cd} X_{ef}
\]
\[
= 2t^2 F^{\alpha\beta} B^{\alpha c} B^{\beta f} B^{\beta\alpha} X_{cd} X_{ef} - 4t F^{\alpha\beta} B^{\alpha c} B^{\beta f} X_{cd} X_{n+1} + 2F^{\alpha\beta} B^{\alpha\beta} X_{n+1}^2
\]
by breaking the summation into the following three parts.

**Case 1.** \( \alpha, \beta \in T \). We can see that if \( c = n \), then \( d \) must be \( n+1 \), as \( B^{n+1\beta} = 0 \), and
\[
(2.14) \quad 2 \sum_{i \in T} t^2 B^{\alpha\beta} B^{\alpha c} B^{\beta\beta} X_{\alpha i} X_{\beta i} - 4t B^{\alpha\beta} B^{\alpha c} B^{\beta\beta} X_{\alpha i} X_{n+1} + 2F^{\alpha\beta} B^{\alpha\beta} X_{n+1}^2
\]
\[
= \sum_{i \in T} \frac{2}{B^{\alpha\beta}} (t B^{\alpha\beta} B^{\alpha c} X_{\alpha i} - B^{\alpha\beta} X_{n+1}) (t B^{\alpha\beta} B^{\alpha c} X_{\beta i} - B^{\alpha\beta} X_{n+1}).
\]

**Case 2.** \( \alpha = n, \beta \in T \) or \( \beta = n, \alpha \in T \). As \( B^{\alpha\beta} = \frac{1}{t} \), we have
\[
(2.15) \quad 2 \sum_{i \in T} t^2 B^{\alpha\beta} B^{\alpha c} B^{\beta\beta} X_{\alpha i} X_{\beta i} - 4t B^{\alpha\beta} B^{\alpha c} B^{\beta\beta} X_{\alpha i} X_{n+1} + 2F^{\alpha\beta} B^{\alpha\beta} X_{n+1}^2
\]
\[
= 2t^2 B^{\alpha\beta} B^{\alpha c} (B^{\beta\beta} X_{\alpha i} + B^{\alpha\beta} X_{n+1}) X_{ef}
\]
\[
- 4t B^{\alpha\beta} (B^{\beta\beta} X_{\alpha i} + B^{\alpha\beta} X_{n+1}) X_{n+1} + 2B^{\alpha\beta} X_{n+1}^2
\]
\[
= 2t^2 B^{\alpha\beta} (\sum_{c + n} B^{\alpha c} B^{\beta\beta} X_{\alpha i} + B^{\alpha\beta} X_{n+1}) X_{ef}
\]
\[
- 4t (\sum_{c + n} B^{\alpha\beta} B^{\beta\beta} X_{\alpha i} + B^{\alpha\beta} X_{n+1}) X_{n+1} + 2B^{\alpha\beta} X_{n+1}^2
\]
\[
= 2t^2 B^{\alpha\beta} \sum_{c + n} B^{\alpha c} B^{\beta\beta} X_{\alpha i} X_{ef} - 4t \sum_{c + n} B^{\alpha\beta} B^{\beta\beta} X_{\alpha i} X_{n+1}
\]
\[
= 2t^2 B^{\alpha\beta} \left( \sum_{c + n} B^{\alpha c} B^{\beta\beta} X_{\alpha i} X_{ef} + B^{\alpha\beta} X_{n+1} \sum_{c + n} B^{\alpha\beta} B^{\beta\beta} X_{\alpha i} X_{n+1} \right)
\]
\[
- 4t \sum_{c + n} B^{\alpha\beta} B^{\beta\beta} X_{\alpha i} X_{n+1}
\]
\[
= 2t^2 \sum_{c + n} B^{\alpha\beta} B^{\beta\beta} X_{\alpha i} X_{ef} - 2t \sum_{c + n} B^{\alpha\beta} B^{\beta\beta} X_{\alpha i} X_{n+1}
\]
\[
= 2 \sum_{i \in T} \frac{1}{B^{\alpha\beta}} (t B^{\alpha\beta} B^{\beta\beta} X_{\alpha i} - B^{\alpha\beta} X_{n+1}) \left( t \sum_{c + n} B^{\alpha\beta} B^{\beta\beta} X_{\alpha i} X_{ef} \right),
\]
Case 3. $\alpha = \beta = n$. In this case we deduce

$$
\frac{\partial^2 Q_{mn}}{\partial B_{cd} \partial B_{ef}} X_{cd} X_{ef} = 4t^2 \sum_{c,d,n} (B_{mn} B_{n+1} B_{dn} X_{m+1} + B_{mn} B_{n+1} B_{dn} X_{m+1}) \\
+ 2t^2 \sum_{c,d,e,f,n} B_{ne} B_{ce} B_{df} X_{cd} X_{ef} \\
+ 2t^2 B_{mn} (B_{n+1} B_{n+1} X_{n+1} + B_{n+1} B_{n+1} X_{n+1} + B_{n+1} B_{n+1} X_{n+1}) \\
- 4t \sum_{c,d,n} B_{ne} B_{ce} X_{n+1} - 2t B_{mn} B_{n+1} X_{n+1} + 2B_{mn} X_{n+1}^2.
$$

From the fact that $B_{n+1} = \frac{1}{t}$ and $B_{n+1} = 0$ for all $\beta \neq n$, it follows that

$$
(2.16) \quad F_{mn} \frac{\partial^2 Q_{mn}}{\partial B_{cd} \partial B_{ef}} X_{cd} X_{ef} = 2t^2 \sum_{c,d,e,f,n} F_{mn} B_{ne} B_{ce} B_{df} X_{cd} X_{ef} \\
= 2t^2 \sum_{i \in T, d,e,f,n} F_{mn} B_{ne} B_{ei} X_{cd} X_{ef} \\
= 2 \sum_{i \in T} \frac{F_{mn}}{B_{ii}} \left( t \sum_{e+n} B_{ne} B_{ei} X_{ei} \right)^2.
$$

Set

$$
Y_{ix} := t B_{ii} B_{xx} X_{ix} - B_{xx} X_{x+1}, \quad \forall x \in T,
$$

$$
(2.17) \quad Y_{in} := t B_{ii} \sum_{c+n} B_{ne} X_{ci}.
$$

Combining (2.14), (2.15) and (2.16), for $\theta = (0, \ldots, 0, 1)$, $I$ in (2.5) can be written as

$$
(2.18) \quad I = 2 \sum_{i \in T} \frac{F_{x\beta}}{B_{ii}} Y_{ix} Y_{i\beta},
$$

where $Y_{ix}$ for all $x \in \{1, \ldots, n\}$ is defined in (2.17).

We wish to express $I$ in terms of $\hat{X}_{x\beta}$ and $A$. Recall $Q_{x\beta} = t^2 B_{x\beta}$.

$$
\frac{\partial Q_{x\beta}}{\partial B_{cd}} = \frac{\partial (t^2 B_{x\beta})}{\partial B_{cd}} = -t^2 B_{x\beta} B_{x\beta} + 2 t B_{x\beta} \delta_{x+n, \delta_{x+n+1}} \\
\sum_{c,d} t \frac{\partial Q_{x\beta}}{\partial B_{cd}} X_{cd} - 3Q_{x\beta} \sum_{c,d} X_{n+1} \theta = -t^3 \sum_{c,d} B_{x\beta} B_{x\beta} X_{cd} - t^2 B_{x\beta} X_{n+1}.
$$
By (2.9)

\[ t^2 \dot{\mathbf{X}}_{ab} = -t \mathbf{B}^{ce} \mathbf{B}^{df} \mathbf{X}_{cd} - \mathbf{B}^{df} \mathbf{X}_{n+1,n}, \quad 1 \leq c, d \leq n. \]  

We extend the definition of \( \dot{\mathbf{X}} \) as \( (n + 1) \times (n + 1) \) symmetric matrix by setting

\[ t^2 \dot{\mathbf{X}}_{ef} = -t \mathbf{B}^{ce} \mathbf{B}^{df} \mathbf{X}_{cd} - \mathbf{B}^{df} \mathbf{X}_{n+1,n}, \quad 1 \leq e, f \leq n + 1. \]

Since \( \mathbf{X}_{ax} = 0 \) for all \( a \leq n \), and \( \mathbf{B}^{n+1,c} = \mathbf{A}_{n+1,c} = 0 \) for all \( c + n \), we have

\[ \dot{\mathbf{X}}_{n+1,c} = 0, \quad \forall c \neq n; \quad \ddot{\mathbf{X}}_{n+1,n} = -2 \frac{t}{t^3} \mathbf{X}_{n+1,n} = \frac{2}{t} \mathbf{\bar{Y}}. \]

In this setting, \( \mathbf{X}_{cd} \) can be recovered using the formula

\[ \mathbf{X}_{cd} = \frac{t^2 A^{ce} A^{df}}{2} \dot{\mathbf{X}}_{ef} + \frac{t^2 A^{cd}}{2} \ddot{\mathbf{X}}_{n+1,n}, \quad 1 \leq c, d \leq n + 1. \]

From the relationship \( \mathbf{B} = \mathbf{A}^{-1} \), and the fact that the \( (n - 1) \times (n - 1) \) matrix \( \mathbf{A}_{ab} \) is diagonal, we have \( \mathbf{B}_{n+1,i} = -\frac{t A_{n,i}}{A_{ii}} \). Hence from (2.17), it follows that

\[ Y_{is} = t^3 A_{is} \ddot{\mathbf{X}}_{n+1,n} - t^2 \dot{\mathbf{X}}_{is}. \]

From (2.18) and (2.23) we can deduce the next lemma.

**Lemma 2.5.** For \( \theta = (0, \ldots, 0, 1) \), if \( \mathbf{A}_{ab} \) is diagonal, then \( I \) in (2.5) can be written as

\[ I = 2 \sum_{i \in T} \frac{F_{i\beta}}{A_{ii}} Y_{is} Y_{i\beta}, \]

where \( Y_{is} \) is defined in (2.23).

From (2.6), (2.7) and (2.9), we get

\[ \langle V, \nabla_{(B,x)} F \rangle = F_{i\beta} \dot{\mathbf{X}}_{i\beta} + F_{ik} \mathbf{\bar{Y}} + F_{ik} \mathbf{Z}_k \]

and we have \( V = ((X_{cd}), (Z_k)) \in T \Gamma_T \) if and only if

\[ F_{i\beta} \dot{\mathbf{X}}_{i\beta} + F_{ik} \mathbf{\bar{Y}} + F_{ik} \mathbf{Z}_k = 0, \]

where \( \mathbf{X}_{i\beta}, \mathbf{\bar{Y}} \) are as in (2.19) and (2.21), respectively, and \( F_{i\beta}, F_{ik} \) and \( F_{ik} \) are evaluated at \( (t^{-1} \mathbf{A}, t^{-1} \theta, u, x) \) with \( \theta = (0, \ldots, 0, 1) \).

Set \( \mathbf{\bar{V}} = ((\mathbf{X}_{cd}), (Z_k)) \) where \( (\mathbf{X}_{cd}) \) is defined by (2.20) with (2.21). Rewrite (2.10) as

\[ H(\mathbf{\bar{V}}, \mathbf{\bar{V}}) = \frac{I}{t^3} + S, \]
where \( I \) is defined as in (2.24),

\[
(2.27) \quad S = F^{ab,\beta,\gamma} \hat{X}_{ab} \hat{Y} + 2F^{a\beta,\gamma\delta} \hat{X}_{ab} \hat{Y} + 2F^{a\beta,\gamma\delta} \hat{X}_{ab} \hat{Z}_k + F^{a\beta,\gamma\delta} \hat{Y}^2 + 2F^{a\beta,\gamma\delta} \hat{Z}_k \hat{Y}_k + 2F^{a\beta,\gamma\delta} \hat{Z}_k \hat{Y}_k + 6F^{a\beta,\gamma\delta} \hat{X}_{ab} \hat{Y}^2 - 6F^{a\beta,\gamma\delta} \hat{X}_{ab} \hat{Y}^2,
\]

and \( F^{ab}, F^{u\alpha}, F^{x\gamma} \) etc. are evaluated at \((t^{-1} \hat{A}, t^{-1} \theta, u, x), \hat{A} \in \mathcal{S}_n^\perp(\theta) \cap \mathcal{Y} \).

By (2.22) and Lemma 2.3, we get the following result.

**Lemma 2.6.** Condition (1.8) is equivalent to

\[
H(\hat{V}, \hat{V}) \leq 0, \quad \forall \hat{V} = ((\hat{X}_{\alpha\beta}), (Z_k)) \text{ satisfying (2.25)}.
\]

By approximation, if condition (1.8) is satisfied, then

\[
H(\hat{V}, \hat{V}) \leq 0
\]

for every \( \hat{V} = ((\hat{X}_{\alpha\beta}), (Z_k)) \) satisfying (2.25) at each \( \hat{A} \in \mathcal{S}_n^\perp(\theta) \cap \mathcal{Y} \) diagonal, where \( \mathcal{S}_n^\perp(\theta) \) is the closure of \( \mathcal{S}_n^\perp(\theta) \), with \( Y_{k\alpha} = 0 \) when \( a_k = 0 \) for some \( k \leq n - 1 \).

With the explicit expression of \( H \) in Lemma 2.6, we may verify condition (1.8) for the mean curvature operator and the general quasilinear operator \( F \) satisfying the structural conditions in [14]. Condition (1.6) is not satisfied by the mean curvature operator as indicated in [3]. It was verified there that for \( n = 2 \), the mean curvature operator

\[
(2.28) \quad F(D^2u, Du) = \text{Div} \left( \frac{Du}{\sqrt{1 + |Du|^2}} \right) = \frac{\Delta u}{\sqrt{1 + |Du|^2}} - \sum_{a, b = 1}^n \frac{u_a u_b u_{ab}}{\sqrt{1 + |Du|^2}}
\]

satisfies condition (1.8), but not (1.6). Here we verify this fact for general \( n \). Since condition (1.8) and (1.6) are invariant under orthogonal transformation, we may as well set \( Du = (0, \ldots, 0, u_\theta) \), \( (u_\theta) \) is diagonal for each \( i, j \in T = \{1, \ldots, n - 1\} \). We also note that \( (u_\theta) \) is negative definite. According to (1.4), we have \( t^{-1} \theta = Du \), where \( \theta = (0, \ldots, 0, 1) \) and \( t^{-1} = u_\mu \), \( J B^{-1} J^T = u_\mu^{-1}(D^2u) \). Since the mean curvature operator \( F \) in (2.28) is homogenous of one degree, \( S \) in (2.27) can be calculated as

\[
(2.29) \quad S = 2F^{a\beta, u\alpha} \hat{X}_{ab} \hat{Y} + F^{u\alpha, u\beta} \hat{Y}^2 + 2F^{u\alpha, u\beta} \hat{Y}^2 + 2F^{u\alpha, u\beta} \hat{Z}_k \hat{Y} - 6u^{-2} F \hat{Y}^2.
\]

By (2.25), \( \hat{V} = (\hat{X}_{\alpha\beta}) \) satisfies

\[
(2.30) \quad 0 = \langle \hat{V}, \nabla_B F \rangle = F^{a\beta} \hat{X}_{ab} + F^{u\alpha} \hat{Y}.
\]

A straightforward calculation yields that

\[
(2.31) \quad S = \sum_{i \in T} \left( \frac{4u_{a_i}}{W^3} \hat{X}_{a_i} \hat{Y} - \frac{6}{W^3} u_{a_i} \hat{Y}^2 \right) - \frac{3}{W^2} u_{a_i}^2 F \hat{Y}^2,
\]
where \( W = \sqrt{1 + |Du|^2} \). It is easy to check that \( S \leq 0 \) is violated for some \( \bar{X}_i, \bar{Y} \) satisfying (2.30). On the other hand, [2], Lemma 4.1, implies that \( S \leq 0 \) if condition (1.6) is satisfied. Therefore, \( F \) does not satisfy condition (1.6).

However, from (2.24)

\[(2.32)\quad I = 2u_n \sum_{i \in T} \frac{F^{\beta \beta}}{u_{\beta}} Y_{ix} Y_{\beta \beta}.\]

For the mean curvature equation, it can be computed that

\[(2.33)\quad Y_{ix} = -t^2 \bar{X}_{ix} + 2t^2 u_{ix} \bar{Y} u_n^{-1}, \quad \forall i, x \in G.\]

By (2.31), (2.32), (2.33) and the facts that \( u_{ii} < 0 \) and \( F \geq 0 \), we deduce

\[
H = S + Iu_n^3 \\
\leq S + 2u_n^4 \sum_{i \in T} \frac{F^{\beta \beta}}{u_{\beta}} Y_{ii}^2 \\
= 2 \sum_{i \in T} \left( \frac{1}{W u_{i i}} \bar{X}_{ii}^2 - 2 \frac{u_n + 2u_n^{-1}}{W^2} \bar{X}_{ii} \bar{Y} + \frac{4u_n^{-2} + 1}{W^3} \bar{u}_{ii} \bar{Y}^2 \right) - \frac{3 + 6u_n^{-2}}{W^4} F \bar{Y}^2 \\
= \sum_{i \in T} \frac{2}{W u_{i i}} \left[ \left( \frac{1}{W^2} u_n + 2u_n^{-1} u_{i i} \bar{Y} \right)^2 + \frac{1}{W^4} u_{ii} \bar{Y}^2 \right] - \frac{3 + 6u_n^{-2}}{W^4} F \bar{Y}^2 \\
\leq 0.
\]

That is, the mean curvature operator \( F \) satisfies condition (1.8) by Lemma 2.6. This example indicates that the term \( I \) is the key. The verification of condition (1.8) for the quasilinear operators considered in [11], [14] can be done in a similar way, which we leave for the interested reader.

3. The test function

The proof of our main results relies on the establishment of a maximum principle for a certain appropriate curvature test function. This section is devoted to discuss some regularity and concavity properties of the proposed test function.

In the remainder of the paper, we assume that \( u \in C^{3,1}(\Omega) \), \( |\nabla u| > 0 \) and that \( \{ x \in \Omega : u(x) \geq c \} \cap \Omega_1 \) is locally convex.

We recall some formulas related to the Weingarten curvature tensor of level surfaces. Suppose \( u \) is a function defined in an open set in \( \mathbb{R}^n \), assume that \( u_n(x) \neq 0 \). The upward inner normal direction of the level sets of \( u \) is

\[(3.1)\quad \bar{\nu} = \frac{|u_n|}{|Du|} (u_1, u_2, \ldots, u_{n-1}, u_n).\]
It is calculated in [2] that the second fundamental form $II$ of the level surface of function $u$ with respect to the upward normal direction (3.1) is

$$h_{ij} = -\frac{|u_n|(u^2_{ij} + u_j u_i)}{|Du|^2}, \quad i, j \leq n - 1.$$  

(3.2)

Note that as $\{x \in \Omega : u(x) \geq c\} \cup \Omega_1$ is locally convex, the second fundamental form of $\Sigma'$ is nonnegative definite with respect to the upward normal direction (3.1). Since $Du$ is the same direction as $\tilde{n}$, we have $u_n > 0$ locally. (3.2) implies that the matrix $(u_j(x))$ is nonpositive definite.

Denote $a(x) = (a_j(x))$ the symmetric Weingarten tensor of

$$\Sigma''(x) = \{y \in \Omega : u(y) = u(x)\}.$$

Our assumption implies that $a$ is nonnegative definite. Since $u_n(x) \neq 0$, following [4], the Weingarten tensor can be computed as (see [2])

$$a_{ij} = h_{ij} - \frac{\sum_{l=1}^{n-1} u_l u_l h_{ij} - \sum_{l=1}^{n-1} u_l u_l h_{ij} + \sum_{k=1}^{n-1} u_l u_k u_k h_{jl}}{W(1 + W)^2 u_n^2 + W^2(1 + W)^2 u_n^4}, \quad i, j \leq n - 1,$$

(3.3)

where $W = (1 + |\nabla_x v|^2)^{\frac{1}{2}}$ and $x' = (x_1, \ldots, x_{n-1})$.

Set

$$\tilde{a} = a - \eta_0 g(u)I, \quad \eta_0 \geq 0, \quad g(u) = e^{\alpha u},$$

where $\eta_0 \geq 0$ and $A \geq 0$ are constants to be determined later such that $\tilde{a} \geq 0$.

Suppose the minimal rank $l$ of $\tilde{a}$ is attained at some interior point $x_0$. Let $\mathcal{O}$ be a small open neighborhood of $x_0$ such that for each $x \in \mathcal{O}$, there are $l$ “good” eigenvalues of $(\tilde{a}_j)$ which are bounded below by a positive constant, and the other $n - 1 - l$ “bad” eigenvalues of $(\tilde{a}_j)$ are very small. Denote by $G$ the index set of these “good” eigenvalues and by $B$ the index set of “bad” eigenvalues. For each $x \in \mathcal{O}$ fixed, we may express $(a_j)$ in a form of (3.3), by choosing $e_1, \ldots, e_{n-1}, e_n$ such that

$$|Du|(x) = u_n(x) > 0, \quad (a_j(x)), i, j = 1, \ldots, n - 1, \text{ is diagonal.}$$

(3.5)

From (3.3) and (3.4), the matrix $(\tilde{a}_j)$, $i, j = 1, \ldots, n - 1$, is also diagonal at $x$, and without loss of generality we may assume $\tilde{a}_{11} \leq \tilde{a}_{22} \leq \cdots \leq \tilde{a}_{n-1,n-1}$. There is a positive constant $C > 0$ depending only on $\|u\|_C^1$ and $\mathcal{O}$, such that $\tilde{a}_{n-1,n-1} \geq \tilde{a}_{n-2,n-2} \geq \cdots \geq \tilde{a}_{n-1,n-1} > C$ for all $x \in \mathcal{O}$. For convenience we denote by $G = \{n - l, n - l + 1, \ldots, n - 1\}$ and $B = \{1, 2, \ldots, n - l - 1\}$ the “good” and “bad” sets of indices, respectively. If there is no confusion, we also write

$$B = \{\tilde{a}_{11}, \ldots, \tilde{a}_{n-l-1,n-l-1}\} \quad \text{and} \quad G = \{\tilde{a}_{n-l,n-l}, \ldots, \tilde{a}_{n-1,n-1}\}.$$
Note that for any \( \delta > 0 \), we may choose \( \mathcal{O} \) small enough such that \( \tilde{a}_{ij} < \delta \) for all \( j \in B \) and \( x \in \mathcal{O} \).

The following two functions are of fundamental importance in our treatment:

\[
(3.7) \quad p(\tilde{a}) = \sigma_{l+1}(\tilde{a}_{ij}), \quad q(\tilde{a}) = \begin{cases} \frac{\sigma_{l+2}(\tilde{a}_{ij})}{\sigma_{l+1}(\tilde{a}_{ij})}, & \text{if } \sigma_{l+1}(\tilde{a}_{ij}) > 0, \\ 0, & \text{otherwise}. \end{cases}
\]

We consider the function

\[
(3.8) \quad \phi(\tilde{a}) = p(\tilde{a}) + q(\tilde{a}),
\]

where \( p \) and \( q \) are as in (3.7). The function \( \phi \) was first introduced in [1] for the Hessian of the solution \( u \), and for the Weingarten tensor \( a \) in [2]. Here we adopt it as a function in \( \tilde{a} \).

We will use the notion \( h = O(f) \) if \( |h(x)| \leq Cf(x) \) for \( x \in \mathcal{O} \) with positive constant \( C \) under control. Again, as in [1], to get around \( p = 0 \), for \( \varepsilon > 0 \) sufficiently small, we consider instead

\[
(3.9) \quad \phi_\varepsilon(\tilde{a}) = \phi(\tilde{a} + \varepsilon I),
\]

where \( \tilde{a}_\varepsilon = \tilde{a} + \varepsilon I \). We will also denote \( G_\varepsilon = \{ \tilde{a}_{ii} + \varepsilon : i \in G \} \), \( B_\varepsilon = \{ \tilde{a}_{ii} + \varepsilon : i \in B \} \).

We will write \( p \) for \( p_\varepsilon \), \( \phi \) for \( \phi_\varepsilon \), \( q \) for \( q_\varepsilon \), \( \tilde{a} \) for \( \tilde{a}_\varepsilon \), \( G \) for \( G_\varepsilon \), \( B \) for \( B_\varepsilon \), with the understanding that all the estimates will be independent of \( \varepsilon \). In this setting, if we pick \( \mathcal{O} \) small enough, there is \( C > 0 \) independent of \( \varepsilon \) such that

\[
(3.10) \quad \phi(\tilde{a}(z)) \geq C\varepsilon, \quad \sigma_1(B(z)) \geq C\varepsilon, \quad \forall z \in \mathcal{O}.
\]

In what follows, \( i, j, \ldots \) will denote indices running from 1 to \( n \) and the Greek indices \( \alpha, \beta, \ldots \) will denote indices from 1 to \( n \). We write

\[
\begin{align*}
p_x &= \frac{\partial p}{\partial x^i}, \\
p_{\alpha\beta} &= \frac{\partial^2 p}{\partial x^i \partial x^\beta}, \\
F_{\alpha\beta} &= \frac{\partial F}{\partial u_{\alpha\beta}}, \quad 1 \leq \alpha, \beta \leq n.
\end{align*}
\]

We also denote \( g = e^{Au} \).

\[
(3.11) \quad \mathcal{H}_\phi = \sum_{i, j \in B} |\nabla \tilde{a}_{ij}| + \phi,
\]

and for all \( j \in B \),

\[
(3.12) \quad I_j = \sum_{i \in G} \left[ -2u^3 \sum_{x, \beta \in B} F^{x\beta} a_{ij} a_{i\beta} \right] + 4u^2 u_{ij} \sum_{x \in B} F^{xj} a_{ij} + 2u^2 F^{ii} u_{ij},
\]
and

\[ J_{ij} = -12 \sum_{x=1}^{n} F^{xj} u_{nx} u_{nx} - 4 \sum_{x=1}^{n} F^{xj} u_{mn} u_{nj} - 2 u_{nm} F^{ii} u_{ij} \]

\[ - \eta_{0} \sum_{x, \beta = 1}^{n} F^{x\beta} g_{x\beta} u_{n}^{3} + 4 \eta_{0} \sum_{x=1}^{n} F^{xj} g_{xj} u_{n}^{2} - 2 \eta_{0} \sum_{x, \beta = 1}^{n} F^{x\beta} u_{nx} g_{j} u_{n} \]

\[ - 2 \eta_{0} g \sum_{x, \beta = 1}^{n} \sum_{i, n} F^{x\beta} a_{ij} a_{ii} u_{n}^{3}, \]

and

\[ J_{ij} = 2 F^{x\beta} u_{xj} u_{ij} + 2 F^{u_{ij}} u_{xj} + 2 F^{u_{ij}} + u_{ij} \]

\[ + F^{u_{ij}} u_{ij} + 2 F^{u_{ij}} - \eta_{0} F^{u_{ij}} u_{ij}. \]

**Lemma 3.1.** Suppose \( u \in C^{3,1} \) is a solution of equation (1.1) with \( |\nabla u| > 0 \), then \( \phi \in C^{1,1}(\mathcal{O}) \). For any fixed \( x \in \mathcal{O} \), with the coordinate chosen as in (3.5) and (3.6),

\[ \phi_{x} = \left[ \sigma_{1}(G) + \frac{\sigma_{1}^{2}(B | j) - \sigma_{2}(B | j)}{\sigma_{1}^{2}(B)} \right] \tilde{a}_{ij} + O(\mathcal{H}_{\phi}) \]

and

\[ F^{x\beta} \phi_{x\beta} = \sum_{j=1}^{n} u_{n}^{-3} \left[ \sigma_{i}(G) + \frac{\sigma_{i}^{2}(B | j) - \sigma_{2}(B | j)}{\sigma_{i}^{2}(B)} \right] \]

\[ \times \left\{ \sum_{x, \beta, \gamma, \eta = 1}^{n} F^{x\beta} y_{x\eta} y_{x\gamma} y_{x\gamma} + 2 \sum_{x, \beta = 1}^{n} F^{x\beta} y_{xj} y_{xj} + 2 \sum_{x, \beta = 1}^{n} F^{x\beta} u_{xj} \right\} \]

\[ + 6 \sum_{x, \beta = 1}^{n} F^{x\beta} y_{xj} y_{xj} - 6 \sum_{x, \beta = 1}^{n} F^{x\beta} u_{xj} u_{n} + I_{j} + J_{ij} + J_{2j} \]

\[ - \frac{1}{\sigma_{1}(B)} \sum_{x, \beta = 1}^{n} \sum_{j=1}^{B} F^{x\beta} \left[ \sigma_{1}(B) \tilde{a}_{ij} - \tilde{a}_{ij} \right] \left[ \sigma_{1}(B) \tilde{a}_{ij} - \tilde{a}_{ij} \right] \]

\[ - \frac{1}{\sigma_{1}(B)} \sum_{x, \beta = 1}^{n} \sum_{j=1}^{B} F^{x\beta} \tilde{a}_{ij} \tilde{a}_{ij} + O(\mathcal{H}_{\phi}). \]

**Proof.** For any fixed point \( x \in \mathcal{O} \), choose a coordinate system as in (3.5) so that \( |Du(x)| = u_{x}(x) > 0 \) and the matrix \( \left( \tilde{a}_{ij}(x) \right) \) is diagonal for \( 1 \leq i, j \leq n - 1 \) and
nonnegative. From the definition of $p$,

\[
\frac{u_{ij}}{u_n} - \eta_0 g = \bar{a}_{ij} + O(\mathcal{H}_\phi), \quad \forall j \in B; \quad p_x = \sigma_l(G) \sum_{j \in B} \bar{a}_{ij, x} + O(\phi).
\]

By (3.17),

\[
p_{x\beta} = \sigma_l(G) \left[ \sum_{j \in B} \bar{a}_{ij, \beta} - 2 \sum_{i \in G, j \in B} \frac{\bar{a}_{ij, x} \bar{a}_{ij, \beta}}{\bar{a}_{ij}} \right] + O(\mathcal{H}_\phi) = \sigma_l(G) \left[ \sum_{j \in B} (a_{ij, \beta} - \eta_0 g_{x\beta}) - 2 \sum_{i \in G, j \in B} \frac{\bar{a}_{ij, x} \bar{a}_{ij, \beta}}{\bar{a}_{ij}} \right] + O(\mathcal{H}_\phi).
\]

Since $u_k = 0$ at $x$ for $k = 1, \ldots, n - 1$, from (3.3), and for each $j \in B$,

\[
u_n^3 a_{ij, x\beta} = -u_n^2 u_{ij\beta} - 2u_n (u_{ij\beta} u_{ij} + u_{ij} u_{ij\beta}) + 2u_n (u_{ij\beta} u_{ij} + u_{ij\beta} u_{ij\beta})
+ 2u_n u_{ij\beta} u_{ij} + 2u_n (u_{ij} u_{ij\beta} + u_{ij\beta} u_{ij}) - 2u_n u_{ij\beta} u_{ij}\beta
- (2u_{ij\beta} + 2u_n u_{ij\beta}) u_{ij} - 2\eta_0 u_{ij\beta} u_{ij\beta} u_n - 3\eta_0 u_n^2 (u_{ij\beta} + u_{ij\beta} u_{ij})
- \eta_0 g \left( 3u_n^2 u_{ij\beta} + 6u_n u_{ij\beta} u_n + \sum_{i=1}^{n-1} u_{ij} u_{ij\beta} u_n \right) + O(\mathcal{H}_\phi).
\]

From the definition of $a_{ij}$, we deduce

\[
u_n u_{ijx} = -u_n^2 a_{ij, x} + u_{ij} u_{ix} + u_{ix} u_{ij}, \quad \forall i, j \leq n - 1,
\]

\[
\bar{a}_{ij, x} = a_{ijx} - \eta_0 g_{x\beta} \delta_{ij}, \quad \forall i, j \in B,
\]

and

\[
\sum_{x, \beta = 1}^{n} F_{x\beta}^{x\beta} a_{ij, x\beta} = \sum_{x, \beta = 1}^{n} \frac{F_{x\beta}^{x\beta}}{u_n} \left[ -u_n^2 u_{ijx\beta} - 4u_n u_{ij\beta} u_{ij} + 4u_n u_{ijx} u_{ij\beta}
+ 2u_n u_{ij\beta} u_{ij} - 2u_n u_{ij\beta} u_{ij}\beta
- \eta_0 g \left( 2u_{ij} u_{ij\beta} u_n + u_{ijx} u_n^2 + \sum_{i=1}^{n-1} u_{ij} u_{ij\beta} u_n \right)
- 2\eta_0 u_{ijx} u_{ij\beta} u_n^2 \right] + O(\mathcal{H}_\phi).
\]

Breaking the summation as

\[
\sum_{x=1}^{n} F_{x\beta}^{x\beta} u_{ijx} = \left( \sum_{x, \beta = 1}^{n} - \sum_{x=1}^{n} \sum_{\beta=1}^{n} \right) F_{x\beta}^{x\beta} u_{ij\beta},
\]
and for all \( j \in B \) similarly
\[
\sum_{x, \beta = 1}^{n} F^{\beta} u_{2x} u_{\beta j} = u_{nj} \left( \sum_{x, \beta = 1}^{n} - \sum_{x = 1}^{n-1} \sum_{\beta = 1}^{n} \right) F^{\beta} u_{2x} + \sum_{x = 1}^{n} F^{j} u_{2x} u_{jj},
\]
\[
\sum_{x, \beta = 1}^{n} F^{\beta} u_{j2} u_{\beta j} = u_{nj} \left( \sum_{x, \beta = 1}^{n} - \sum_{x = 1}^{n-1} \sum_{\beta = 1}^{n} \right) F^{\beta} u_{j2} + \sum_{x = 1}^{n} F^{j} u_{j2} u_{jj},
\]
by (3.20), for \( j \in B \), we get
\[
u_{n} \sum_{x = 1}^{n} \sum_{\beta = 1}^{n-1} F^{\beta} u_{\beta j} = u_{nj} \left( \sum_{x = 1}^{n} \sum_{i \in G} F^{j} u_{ij} + \sum_{i \in G} F^{j} u_{ij} \right)
\]
\[
= \sum_{x = 1}^{n} \sum_{i \in G} F^{j} u_{ij} + \sum_{i \in G} F^{j} u_{ij} - \eta_{1} \sum_{j = 1}^{n-1} \sum_{i = 1}^{n} F^{j} u_{ij} + O(\mathcal{H}_{\phi})
\]
\[
= -u_{nj} \sum_{x = 1}^{n} \sum_{i \in G} F^{j} a_{ij, x} + \sum_{i \in G} F^{j} u_{ij} + \sum_{i = 1}^{n} \sum_{x = 1}^{n-1} F^{j} u_{ij} - \eta_{1} \sum_{j = 1}^{n-1} \sum_{i = 1}^{n} F^{j} u_{ij} + u_{nj} \sum_{i \in G} F^{j} u_{ij}
\]
and
\[
\sum_{x, \beta = 1}^{n} F^{\beta} u_{j2} u_{\beta j} = F^{m} u_{nj}^{2} + 2F^{j} u_{nj} u_{jj} + F^{j} u_{jj}^{2}
\]
\[
= u_{nj}^{2} \left( \sum_{x, \beta = 1}^{n} F^{\beta} u_{j2} - 2 \sum_{x = 1}^{n-1} F^{m} u_{2x} - \sum_{x, \beta = 1}^{n} F^{\beta} u_{jj} \right)
\]
\[
+ 2F^{j} u_{nj} u_{jj} + F^{j} u_{jj}^{2}.
\]
Putting above to (3.22) gives
\[
(3.23) \quad \nu_{n} \sum_{x, \beta = 1}^{n} F^{\beta} a_{ij, xj} = -u_{nj}^{2} \sum_{x, \beta = 1}^{n} F^{\beta} u_{j2} + 6u_{nj} u_{nj} - 6u_{nj}^{2} u_{nj}^{2}
\]
\[
+ 4u_{nj} u_{nj} \sum_{x = 1}^{n} \sum_{i \in G} F^{j} a_{ij, x} + 2u_{nj}^{2} \sum_{i \in G} F^{j} u_{ij} + 2u_{nj}^{2} \sum_{i \in B} F^{j} u_{ij}
\]
\[
- 12 \sum_{x = 1}^{n} F^{j} u_{2x} u_{nj} u_{jj} + 4 \sum_{x = 1}^{n} F^{j} u_{j2} u_{2x} u_{jj} - 2u_{nj} F^{j} u_{jj}^{2}.
\]
It follows from (3.20) that at $x$

$$\begin{align*}
- \eta_0 g \sum_{x, \beta=1}^{n} F^{2\beta} \left( 2u_{ij} x_{ij} + u_{n2} x_{n2}^2 + \sum_{i=1}^{n-1} u_{i2} x_{i2} + u_{n} \right) \\
- 2\eta_0 \sum_{x, \beta=1}^{n} F^{2\beta} u_{n2} x_{n2} + 4\eta_0 \sum_{x=1}^{n} F^{2j} u_{j2} x_{j2}^2 + O(\mathcal{H}_\phi).
\end{align*}$$

Since $a_{ij,x} = \tilde{a}_{ij,x}$ for $i \neq j$,

$$\begin{equation}
(3.24) \quad \sum_{x, \beta=1}^{n} F^{2\beta} \frac{\tilde{a}_{ij,x} \tilde{a}_{ij,\beta}}{a_{i\beta}} = \sum_{x, \beta=1}^{n} F^{2\beta} \frac{\tilde{a}_{ij,x} a_{ij,\beta}}{a_{i\beta}} = \sum_{x, \beta=1}^{n} F^{2\beta} \frac{\tilde{a}_{ij,x} a_{ij,\beta}}{a_{i\beta}} \left( 1 + \frac{\eta_0 g}{a_{i\beta}} \right).
\end{equation}$$

Formulas (3.18), (3.23) and (3.24) yield that, for each $j \in B$,

$$\begin{align*}
(3.25) \quad F^{2\beta} p_{xj} &= F^{2\beta} \sigma_l(G) \left[ \sum_{j \in B} (a_{ij, xj} - \eta_0 g_{xj}) - 2 \sum_{i \in G, j \in B} \tilde{a}_{ij,x} \tilde{a}_{ij,\beta} \right] + O(\mathcal{H}_\phi) \\
&= \eta_0^{-3} \sum_{j \in B} \sigma_l(G) \left[ - \sum_{x, \beta=1}^{n} F^{2\beta} u_{n} x_{n2} x_{n2j} + 6u_{n} \sum_{x, \beta=1}^{n} F^{2\beta} u_{mn} x_{n2j} \\
&\quad - 6u_{jn}^2 \sum_{x, \beta=1}^{n} F^{2\beta} u_{n} x_{n2j} + I_j + J_{ij} \right] + O(\mathcal{H}_\phi),
\end{align*}$$

where $I_j$, $J_{ij}$ are as in (3.12) and (3.13).

For each $j \in B$, differentiating equation (1.1) in $e_j$ direction at $x$ yields

$$\begin{equation}
(3.26) \quad \sum_{x, \beta=1}^{n} F^{2\beta} u_{x} u_{nj} + F^{u} u_{nj} + F^{u} u_{n} = 0,
\end{equation}$$

and

$$\begin{equation}
(3.27) \quad \sum_{x, \beta=1}^{n} F^{2\beta} u_{n} x_{n2j} = \sum_{x, \beta, \gamma=1}^{n} F^{2\beta} x_{n2j} u_{n2j} u_{n} + 2 \sum_{x, \beta, l=1}^{n} F^{2\beta} u_{n} x_{n2j} u_{n} \\
+ 2 \sum_{x, \beta, l=1}^{n} F^{2\beta} u_{n} x_{n2j} + \sum_{l, \gamma=1}^{n} F^{u} x_{n} u_{nj} x_{n} + F^{u} u_{n} \\
+ \sum_{l, \gamma=1}^{n} F^{u} x_{n} u_{nj} + F^{u} x_{n} + \sum_{l, \gamma=1}^{n} F^{u} u_{n} x_{n2j}.
\end{equation}$$

It follows from (3.20) that at $x$

$$\begin{equation}
(3.28) \quad \sum_{x, \beta=1}^{n} F^{2\beta} u_{n} x_{n2j} = \sum_{x, \beta, \gamma=1}^{n} F^{2\beta} x_{n2j} u_{n2j} u_{n} + 2 \sum_{x, \beta=1}^{n} \left[ F^{2\beta} u_{n} x_{n2j} u_{n} + F^{2\beta} x_{n} x_{n2j} \right] \\
+ F^{u} x_{n} u_{nj}^2 + 2 F^{u} x_{n} u_{nj} + F^{u} x_{n} + 2 F^{u} u_{n} u_{nj} + J_{2j} + O(\mathcal{H}_\phi),
\end{equation}$$

where $J_{2j}$ is defined in (3.14).
Since \( u_{\beta j} = u_{j\beta} \), from (3.25) and (3.28) we get

\[
(3.29) \quad F_{\beta j}^p = \sum_{j \in B} u_{\beta j}^{-3} \sigma_j(G) \left[ \left( \sum_{x, \beta, \gamma, \eta = 1}^n F_{\beta j}^{x, \gamma} u_{\beta j} u_{\gamma j} \right) + 2 \sum_{x, \beta = 1}^n F_{\beta j}^{x} u_{\beta j} u_{m j} \right] + 2 \sum_{x, \beta = 1}^n F_{\beta j}^{x, \gamma} u_{\beta j} + F_{\beta j}^{u_{\beta j}, u_{m j}} + 2 F_{\beta j}^{u_{\beta j}, u_{m j}} u_{m j}^2
\]

\[
+ F_{\beta j}^{x, \gamma} + 2 F_{\beta j}^{u_{\beta j}, u_{m j}} u_{m j}^2 + 6 \sum_{x, \beta = 1}^n F_{\beta j}^{u_{\beta j} u_{\gamma j} u_{m j}}
\]

\[- 6 \sum_{x, \beta = 1}^n F_{\beta j}^{u_{\beta j} u_{m j} u_{n j}} + I_j + J_{ij} + J_{ij} + O(\mathcal{H}_p). \]

The fact that \( q \in C^{1,1}(\mathcal{O}) \) follows from [1], Corollary 2.2. Also by [1], Lemma 2.4, we have

\[
(3.30) \quad q_x = \frac{\partial q}{\partial x} = \sum_{j \in B} \frac{\sigma_j^2(B \mid j) - \sigma_j(B \mid j)}{\sigma_j^1(B)} \tilde{a}_{ij} + O(\mathcal{H}_p),
\]

and

\[
(3.31) \quad q_{\delta j} = \sum_{j \in B} \frac{\sigma_j^2(B \mid j) - \sigma_j(B \mid j)}{\sigma_j^1(B)} \left[ \tilde{a}_{ij} + 2 \sum_{i \in B} \tilde{a}_{ij} \tilde{a}_{ij} \right] - \frac{1}{\sigma_j^1(B)} \sum_{i \in B} \left[ \sigma_j(B) \tilde{a}_{ij} - \tilde{a}_{ij} \sum_{j \in B} \tilde{a}_{ij} \tilde{a}_{ij} \right] - \frac{1}{\sigma_j^1(B)} \sum_{i \in B} \tilde{a}_{ij} \tilde{a}_{ij} + O(\mathcal{H}_p).
\]

Following the same computations as for \( p \), we get

\[
\sum_{x, \beta = 1}^n F_{\beta j}^p q_{\delta j} = \sum_{j \in B} \frac{\sigma_j^2(B \mid j) - \sigma_j(B \mid j)}{\sigma_j^1(B)} \left[ \left( \sum_{x, \beta, \gamma, \eta = 1}^n F_{\beta j}^{x, \gamma} u_{\beta j} u_{\gamma j} \right) + 2 \sum_{x, \beta = 1}^n F_{\beta j}^{x} u_{\beta j} u_{m j} \right] + 2 \sum_{x, \beta = 1}^n F_{\beta j}^{x, \gamma} u_{\beta j} + F_{\beta j}^{u_{\beta j}, u_{m j}} + 2 F_{\beta j}^{u_{\beta j}, u_{m j}} u_{m j}^2
\]

\[
+ F_{\beta j}^{x, \gamma} + 2 F_{\beta j}^{u_{\beta j}, u_{m j}} u_{m j}^2 + 6 \sum_{x, \beta = 1}^n F_{\beta j}^{u_{\beta j} u_{\gamma j} u_{m j}} - 6 \sum_{x, \beta = 1}^n F_{\beta j}^{u_{\beta j} u_{m j} u_{n j}} + I_j + J_{ij} + J_{ij} + O(\mathcal{H}_p),
\]

The proof of the lemma is complete. \( \square \)
4. Proof of the theorems

We want to create a strong maximum principle for \( \phi \) defined in (3.8). That will imply that \( \bar{a} \) defined in (3.4) is of constant rank. Theorem 1.3 corresponds to the case \( \eta_0 = 0 \). To set a stage for the proof of Theorem 1.4, set

\[
(4.1) \quad \Omega = \left\{ x \in \Omega : 0 < \kappa_s(x) < \frac{\lambda}{100} \right\}.
\]

Proposition 4.1. Suppose \( u \in C^{3,1} \) is a quasiconcave solution of equation (1.1) and \( F \) satisfies the assumptions in Theorem 1.3. If the second fundamental form of \( \Sigma \) of solution \( u \) attains minimum rank \( l \) at a certain point \( x_0 \in \Omega \), then there exist a neighborhood \( O \) of \( x_0 \) and a positive constant \( C \) independent of (3.9). We want to establish the differential inequality (4.2) for all the estimates are independent of that (3.5) and (3.6) are satisfied. We will omit the subindex \( f \) that \( u \) defined in (3.4) is of constant rank. Theorem 1.3 corresponds to the case that \( \bar{a} \) defined in (3.4) is semi-positive definite in \( \Omega \) and it attains minimum rank \( l \) at a certain point \( x_0 \in \Omega \), then inequality (4.2) is true for all \( x \in \Omega \).

Proof. Suppose the minimum rank \( l \) of \( \bar{a} \) is attained at an interior point \( x_0 \), and we may assume \( l \leq n - 2 \). Let \( O \) be a small neighborhood of \( x_0 \). Lemma 3.1 and (3.8) imply that \( \phi \in C^{1,1}(O), \phi(x) \geq 0 \) and \( \phi(x_0) = 0 \). For \( \epsilon > 0 \) sufficient small, let \( \phi_{\epsilon} \) be defined as in (3.9). We want to establish the differential inequality (4.2) for \( \phi_{\epsilon} \) with constant \( C \) independent of \( \epsilon \) in \( O \). For each fixed \( x \in O \), choose a local coordinate frame \( e_1, \ldots, e_{n-1}, e_n \) such that (3.5) and (3.6) are satisfied. We will omit the subindex \( \epsilon \) with the understanding that all the estimates are independent of \( \epsilon \). From Lemma 3.1, it follows

\[
(4.3) \quad F^{ij}_{\epsilon} \phi_{\epsilon} = \sum_{j \in B} u_{n-1}^3 \left\{ \sigma_i(G) + \frac{\sigma_i^2(B j)}{\sigma_1(B)} \right\} (S_j + I_j + J_{ij} + J_{2j})

- \frac{1}{\sigma_1(B)} \sum_{x, \beta=1}^n \sum_{i, j \in B} F^{ij}_{\epsilon} \left( \sigma_1(B) \bar{a}_{ij, x} - \bar{a}_i \bar{a}_j \right) \left( \sigma_1(B) \bar{a}_{ij, \beta} - \bar{a}_i \bar{a}_j \right)

- \frac{1}{\sigma_1(B)} \sum_{x, \beta=1}^n \sum_{i, j, j \in B} F^{ij}_{\epsilon} \bar{a}_{ij, x} \bar{a}_{ij, \beta} + O(H_{\phi}),
\]

where \( I_j, J_{ij}, J_{2j} \) are defined in (3.12), (3.13), (3.14) respectively, and,

\[
(4.4) \quad S_j = \left\{ \sum_{x, \beta, \gamma, \eta=1}^n F^{ij}_{\epsilon, \gamma} u_{j2\beta} u_{j2\eta} + \sum_{x, \beta=1}^n F^{ij}_{\epsilon, u_2} u_{j2\beta} u_{j2\eta} + \sum_{x, \beta=1}^n F^{ij}_{\epsilon, u_2} u_{j2\beta} u_{j2\eta} 

+ F^{ij}_{\epsilon, u_2} u_{j2\beta} u_{j2\eta} + 2 F^{ij}_{\epsilon, u_2} u_{j2\beta} u_{j2\eta} + 2 + \frac{F^{ij}_{\epsilon, u_2}}{u_{j2\beta} u_{j2\eta}} \right\} u_{j2\beta} u_{j2\eta}

+ 6 \sum_{x, \beta=1}^n F^{ij}_{\epsilon, u_2} u_{j2\beta} u_{j2\eta} - 6 \sum_{x, \beta=1}^n F^{ij}_{\epsilon, u_2} u_{j2\beta} u_{j2\eta}.
\]
In the coordinate system (3.5),

\[
(4.5) \quad t = u_n^{-1}, \quad D^2 u(x) = t^{-1} \tilde{A}, \quad A_{ij} = tu_{ij} = \frac{u_{ij}}{u_n}, \quad \theta = (0, \ldots, 0, 1).
\]

For each \( j \in B \), set

\[
\tilde{X}_{\alpha \beta} = u_{\alpha \beta} u_n, \quad \forall \alpha, \beta \in G \cup \{n\} \text{ with } (\alpha, \beta) \neq (n, n),
\]

\[
\tilde{X}_{\alpha \beta} = 2u_{\alpha \beta} u_n, \quad \forall \alpha \in B \text{ or } \forall \beta \in B,
\]

\[
(4.6) \quad \tilde{X}_{\mu \nu} = u_{\mu \nu} u_n + \frac{1}{F_{\mu \nu}} \left[ 2 \sum_{\beta \in G \cup \{n\}} F^{\alpha \beta} u_{\alpha \beta} u_n + \sum_{\alpha, \beta \in B} F^{\alpha \beta} u_{\alpha \beta} u_n \right.
\]

\[
- 2 \sum_{\alpha, \beta \in B} F^{\alpha \beta} u_{\alpha \beta} u_n - 4 \sum_{\alpha \in B} F^{\alpha \alpha} u_{\alpha \alpha} u_n + F_{\mu \nu} u_{\mu \nu} u_n \right],
\]

\[
\tilde{Y} = u_{\mu \nu} u_n, \quad Z_k = \delta_{ij} u_n.
\]

For such \( \tilde{V} = ((\tilde{X}_{\alpha \beta}), \tilde{Y}, (Z_k)) \), by (3.26), we have

\[
\langle V, \nabla_{(B, x)} F \rangle = F^{\alpha \beta} \tilde{X}_{\alpha \beta} + F_{\mu \nu} \tilde{Y} + F_{\nu \nu} \tilde{Z}_k
\]

\[
= F^{\alpha \beta} u_{\alpha \beta} u_n + F_{\mu \nu} u_{\mu \nu} u_n + F_{\mu \nu} u_{\mu \nu} u_n + F_{\nu \nu} u_{\nu \nu} u_n = 0.
\]

We need the following lemma.

**Lemma 4.2.** Under the coordinate system (3.5) at \( x \) with \( \tilde{V} \) as in (4.6),

\[
u_3 I = I_j + O(u_j),
\]

where \( I_j \) is defined in (3.12) and \( I \) in (2.24).

**Proof.** Since \( u_{\alpha \beta} = -a_{\alpha \beta} u_n \) for \( i \in G \) and \( (u_{ij}) \), \( i, j = 1, \ldots, n - 1 \), is diagonal at \( x \), we have by (3.12) and (3.20), for each \( j \in B \),

\[
(4.7) \quad I_j = \sum_{i \in G} \left[ -2u_n^3 \sum_{\beta \not\in B} F^{\alpha \beta} u_{\alpha \beta, x} a_{\alpha \beta} \frac{a_{\alpha \beta}}{u_{ij}} + 4u_n^2 u_{ij} \sum_{\beta \not\in B} F^{\alpha \beta} u_{\alpha \beta, x} + 2u_n^2 F^{\alpha \beta} u_{\alpha \beta, x} \right]
\]

\[
= \sum_{i \in G} \sum_{\beta \not\in B} \left[ -4u_n^2 F^{\alpha \beta} u_{\alpha \beta, x} a_{\alpha \beta} \frac{a_{\alpha \beta}}{u_{ij}} + 8u_n u_{ij} \frac{F^{\alpha \beta} u_{\alpha \beta, x} a_{\alpha \beta}}{u_{ij}} + 8u_n a_{\alpha \beta} F^{\alpha \beta} u_{\alpha \beta, x} \frac{a_{\alpha \beta}}{u_{ij}} \right] + O(u_{ij})
\]

\[
= 2 \sum_{i \in G} \sum_{\beta \not\in B} \frac{F^{\alpha \beta}(u_{ij} - 2u_i u_{ij} u_n - 2u_{ij} u_n)}{u_{ij}} + O(u_{ij}).
\]

By (4.5), (2.21), (2.23) and (4.6),

\[
Y_{\alpha i} = 0, \quad \forall i \in B,
\]

\[
Y_{\alpha i} = -(u_{ij} u_n^{-1} - 2u_i u_{ij} u_n^{-2}), \quad \forall i \in G, \not\in B.
\]
Therefore

\[ u_n^3 I = 2 \sum_{i \in G} \sum_{\beta \in B} F_{x_\beta} Y_{\alpha \beta} u_n^4 = I_j + O(u_j). \]

We now finish the proof of Proposition 4.1.

In case of Theorem 1.3, since \( h_0 = 0 \), \( u_{jj} = O(\phi) \) for all \( j \in B \), \( a = \tilde{a} \). (4.3) gives

\[ F_{ab} = \sum_{i \in B} F_{x_\beta} u_{ij}^3 \left( \sigma_i(G) + \frac{\sigma_1^2(B | j) - \sigma_2(B | j)}{\sigma_1^2(B)} (S_j + I_j) \right) \]

\[ - \frac{1}{\sigma_1^2(B)} \sum_{\beta = 1}^{n} \sum_{i \in B} F_{x_\beta} \left[ \sigma_1(B) a_{ij} - a_{ii} \sum_{j \in B} a_{ij} \right] \left[ \sigma_1(B) a_{ij} - a_{ii} \sum_{j \in B} a_{ij} \beta \right] \]

\[ - \frac{1}{\sigma_1^2(B)} \sum_{\beta = 1}^{n} \sum_{i \in B} F_{x_\beta} a_{ij} a_{ij} \beta + O(\phi). \]

In fact for each \( j \in B \), by (4.6) and (3.20),

\[ \tilde{X}_{ab} = u_{ab} h_n + O(\phi), \quad \forall \alpha \in B \text{ or } \forall \beta \in B, \]

\[ \hat{X}_{ab} = u_{ab} h_n, \quad \forall \alpha, \beta \in G \cup \{ n \}, (\alpha, \beta) \neq (n, n), \]

\[ \tilde{X}_{nn} = u_{nn} h_n + O(\phi). \]

Lemma 4.2, condition (1.8) and Lemma 2.6 imply

\[ S_j + I_j = H(\tilde{V}, \tilde{V}) + O(\phi) \leq O(\phi). \]

By the Newton–MacLaurine inequality,

\[ C \geq \sigma_i(G) + \frac{\sigma_1^2(B | j) - \sigma_2(B | j)}{\sigma_1^2(B)} \geq 0. \]

Condition (1.7) implies that there is \( \delta > 0 \) such that

\[ (F_{x_\beta}) \geq \delta I, \quad \forall \alpha \in O. \]

Combining (4.9), (4.11) and (4.12), we get

\[ F_{x_\beta} \geq C \left( \phi + \sum_{i \in B} |\nabla a_{ij}| \right) - \frac{\delta}{\sigma_1(B)} \sum_{i \in B} \sum_{j \in B} a_{ij}^2 \]

\[ - \frac{\delta}{\sigma_1^2(B)} \sum_{i \in B} \sum_{j \in B} \left( \sigma_1(B) a_{ij} - a_{ii} \sum_{j \in B} a_{ij} \right)^2, \]
Finally, by [1], Lemma 3.3, the term \( \sum_{i,j \in B} |\nabla u| \) can be controlled by the rest terms on the right-hand side in (4.13) and \( \phi + |\nabla \phi| \). In conclusion, there exists a positive constant \( C \) independent of \( \varepsilon \) such that

\[
(4.14) \quad \sum_{\alpha, \beta} F^{\alpha \beta} \phi x_{\alpha \beta} \leq C(\phi + |\nabla \phi|).
\]

Proposition 4.1 is verified under the assumptions of Theorem 1.3. Therefore, Theorem 1.3 is proved.

The remainder of the proof deals with the case \( \eta_0 > 0 \). In this case, since \( \kappa_s(x) > 0 \) for all \( x \in \Omega \) by Theorem 1.3 and since \( \mathcal{O} \subset \Omega \), we have

\[
(4.15) \quad 0 \leq \kappa_s(x) \leq \frac{\lambda}{100 \pi}, \quad \forall x \in \mathcal{O}.
\]

For each \( j \in B \), it follows from (4.6), (3.20), (3.21), and the fact \( u_k = 0 \) for all \( k \leq n - 1 \), that

\[
(4.16) \quad \begin{align*}
\dot{X}_{\alpha \beta} &= u_{\alpha \beta} u_n + a_{\alpha \beta} u_n^2 + \eta_0 g_n u_n^2 \delta_{\alpha \beta} \delta_{j \bar{j}} + O(u_{\bar{j}}), \quad \forall z \in B, \\
\dot{X}_{\alpha \beta} &= u_{\alpha \beta} u_n, \quad \forall z, \beta \in G \cup \{n\}, \ (z, \beta) \neq (n, n), \\
\ddot{X}_{mj} &= u_{mj} u_n - 2 \eta_0 g_n u_n^2 F_{mj} + O(\sum_{i, j \in B} |\nabla u_{\bar{j}}|) + O(u_{\bar{j}}).
\end{align*}
\]

In view of (4.16),

\[
(4.17) \quad \begin{align*}
u_{\alpha \beta} u_n &= \dot{X}_{\alpha \beta} - a_{\alpha \beta} u_n^2 - \eta_0 g_n u_n^2 \delta_{\alpha \beta} \delta_{j \bar{j}} + O(u_{\bar{j}}), \quad \forall z \in B, \\
u_{\alpha \beta} u_n &= \dot{X}_{\alpha \beta}, \quad \forall z, \beta \in G \cup \{n\}, \ (z, \beta) \neq (n, n), \\
u_{mj} u_n &= \ddot{X}_{mj} + 2 \eta_0 g_n u_n^2 F_{mj} + O(\sum_{i, j \in B} |\nabla u_{\bar{j}}|) + O(u_{\bar{j}}).
\end{align*}
\]

Notice that \( u_{\bar{j}} = -\eta_0 g u_n + O(\phi) \) for all \( j \in B \). Substitute \( u_{\alpha \beta} u_n \) by formula (4.17) in \( S_j \), defined in (4.4). We need to track the terms with factor \( \eta_0 g_n^2 \). They are coming from \( \sum_{\alpha, \beta, \gamma, \delta} F_{\alpha \beta \gamma \delta} \), at most, to \( \sum_{\alpha, \beta, \gamma, \delta} F_{\alpha \beta \gamma \delta} \eta_0 g_n^2 u_n^2 \) only. In turn, the coefficient in front of \( \eta_0 g_n^2 \) can be controlled by, say, \( 50 \widetilde{\Omega} \eta_0 g_n^3 \), where \( \widetilde{\Omega} \) is defined in (1.11). By Lemma 4.2, condition (1.8), Lemma 2.6 and the assumptions of Theorem 1.4, there exist constants \( C_0' \), \( C_0'' \) depending only on \( \lambda \), \( \|F\|_{C^2}, d_0, \|u\|_{C^3} \) such that

\[
(4.18) \quad S_j + I_j \leq H(V, V) + 50 \widetilde{\Omega} \eta_0 g_n^3 u_n^3 + C_0' |\eta_0| |g_n| + C_0'' \eta_0 g + O(H_\phi)
\]

\[
\leq 50 \widetilde{\Omega} \eta_0 g_n^3 u_n^3 + C_0' |\eta_0| |g_n| + C_0'' \eta_0 g + O(H_\phi)
\]

\[
= (50 \widetilde{\Omega} \eta_0 g) A^2 \eta_0 g u_n^3 + C_0' A \eta_0 g u_n + C_0'' \eta_0 g + O(H_\phi).
\]
Since $F_{mn} \geq \lambda$, by (3.13) and (3.14),
\begin{equation}
J_{ij} + J_{2j} + O(u_{ij}) \leq -\eta_0 F_{mn} u_m u_n^3 + \eta_0 C_1 g + \eta_0 C_2 g + O(\phi) \\
\leq -\eta_0 \gamma A^2 s_k^3 + \eta_0 g A C_1 u_n + \eta_0 C_2 g + O(\phi),
\end{equation}
where $C_1$, $C_2$, $C_3$ are nonnegative constants depending only on $n$, $d_0$, $\|u\|_{C^1(\Omega)}$, $\|F\|_{C^2}$.

Note that $\eta_0 g \leq \kappa_s(x) < \frac{\lambda}{100\omega}$. It follows from (4.15) that
\begin{equation}
S_j + I_j + J_{ij} + J_{2j} + O(u_{ij}) \leq \frac{\eta_0 g}{2} (A^2 s_k + C_4 A + C_3) + O(\mathcal{H}_d),
\end{equation}
where $C_4$, $C_3$ are nonnegative constants depending only on $n$, $d_0$, $\|u\|_{C^1(\Omega)}$, $\|F\|_{C^2}$. Since $u_n \geq d_0 > 0$, we may choose $A$ large enough in (4.20) depending only on $n$, $d_0$, $\|u\|_{C^1(\Omega)}$, $\|F\|_{C^2}$, $\lambda$ such that
\begin{equation}
S_j + I_j + J_{ij} + J_{2j} + O(u_{ij}) \leq O(\mathcal{H}_d).
\end{equation}

By (4.3) and (4.21),
\begin{equation}
F_{\alpha\beta} a_{\alpha\beta} \leq C \left( \phi + \sum_{i,j \in B} |\nabla a_{ij}| \right) - \frac{\delta}{\sigma_1(B)} \sum_{x=1}^n \sum_{i \in \mathcal{B}} a_{ijx}^2 \\
- \frac{\lambda}{\sigma_1(B)} \sum_{x=1}^n \sum_{i \in \mathcal{B}} \left( \sigma_1(B) a_{ii,x} - a_{ii} \sum_{j \in \mathcal{B}} a_{ij,x} \right)^2.
\end{equation}

As in the case of $\eta_0 = 0$, the same argument yields (4.14) for $\tilde{a}$. Thus Proposition 4.1 is validated under the assumptions in Theorem 1.4. \qed

As pointed out at the beginning of this section, Theorem 1.3 follows directly from Proposition 4.1. We proceed to prove Theorem 1.4.

**Proof of Theorem 1.4.** If $\min\{\kappa^0, \kappa^1\} = 0$, the strict convexity of level surfaces $\Sigma^c$ for $c \in (0, 1)$ in Theorem 1.4 follows from Theorem 1.3. We may assume $\min\{\kappa^0, \kappa^1\} > 0$. By Theorem 1.3, $a$ is strictly positive definite in $\overline{\Omega}$. That is, $\kappa_s(x) > 0$ for all $x \in \overline{\Omega}$. By the continuity, $\kappa_s(x)$ has a positive lower bound (which we need to estimate). With $A \geq 0$ chosen as in the proposition, we increase $\eta_0$ from 0 to the level that $\tilde{a}$ is nonnegative definite throughout $\overline{\Omega}$, but degenerate at some points $x_0$. We have
\begin{equation}
\kappa_s(x) \geq \eta_0 e^{A\tilde{u}(x)}, \quad \kappa_s(x_0) = \eta_0 e^{A\tilde{u}(x_0)}.
\end{equation}

If the degeneracy happens outside of $\Omega$, then $\kappa_s(x_0)$ attains one of the values in $\{\kappa^0, \kappa^1\}$ or $\kappa_s(x_0) \geq \frac{\lambda}{100\omega}$, depending on the location of $x_0$. Consequently, $\eta_0$ attains one of the values in $\{\kappa^0, e^{-A\kappa^1}\}$ or $\eta_0 \geq \frac{\lambda e^{-A\tilde{u}(x_0)}}{100\omega}$, depending on the location of $x_0$. In any case, (1.12) would follow from (4.23).
We need to treat the case when the degeneracy happens at an interior point $x_0$ of $\Omega$. By the proposition, $\bar{a}$ is degenerate with constant rank in a connected component $U$ of $\Omega$ containing $x_0$, and

$$0 = \kappa_s(x) - \eta_0 e^{Au(x)}, \quad \forall x \in U.$$  

By the continuity,

$$0 = \kappa_s(x) - \eta_0 e^{Au(x)}, \quad \forall x \in \bar{U}. \quad \text{(4.24)}$$

That is, the degeneracy of $\bar{a}$ happens also on the boundary of $\Omega$. The case now can be deduced from the previous discussion. At this point, the proof of Theorem 1.4 is complete.

We can say more regarding the geometry of the level surfaces $\Sigma^c$ in the case the degeneracy of $\bar{a}$ happens at an interior point of $\Omega$. If $\bar{U}$ has intersection with the set $\{ \kappa_s(x) = \frac{\lambda}{100 \varepsilon} \}$, choose any $z_0 \in \bar{U} \cap \{ \kappa_s(x) = \frac{\lambda}{100 \varepsilon} \}$, by (4.24),

$$0 = \kappa_s(x) - \eta_0 e^{Au(x)} = \kappa_s(z_0) - \eta_0 e^{Au(z_0)} = \frac{\lambda}{100 \varepsilon} - \eta_0 e^{Au(z_0)}.$$  

This yields

$$\eta_0 = e^{-Au(z_0)} \frac{\lambda}{100 \varepsilon} \quad \text{and} \quad \kappa_s(x) = e^{Au(x) - u(z_0)} \frac{\lambda}{100 \varepsilon}$$

for all $x \in U$. Moreover, the assumption $\kappa_s(x) < \frac{\lambda}{100 \varepsilon}$ for all $x \in U$ forces $u(x) < u(z_0)$ for all $x \in U$. This implies that the level surfaces $\Sigma^c$ will never intersect the set $\{ \kappa_s(x) = \frac{\lambda}{100 \varepsilon} \}$ for any $c \in \left( \inf_{x \in U} u(x), \sup_{x \in U} u(x) \right)$. Note that $\Sigma^c \cap \partial \Omega = \emptyset$ for all $0 < c < 1$, therefore $\Sigma^c$ is a compact hypersurface with constant $\kappa_s(x)$ for $0 < c < u(z_0)$. Since

$$z_0 \in \bar{U} \cap \left\{ \kappa_s(x) = \frac{\lambda}{100 \varepsilon} \right\}$$

is chosen arbitrary, the connect component $U$ of $\Omega$ is exactly the set $\bigcup_{0 < c < c_0} \Sigma^c$ for some $c_0 > 0$ (in fact, $c_0 = u(z_0)$), and $\bar{U} \cap \left\{ \kappa_s(x) = \frac{\lambda}{100 \varepsilon} \right\} = \Sigma^u(z_0)$. Furthermore $\Sigma^c$ is a round sphere for each $0 \leq c \leq c_0$.

If $\bar{U} \cap \left\{ \kappa_s(x) = \frac{\lambda}{100 \varepsilon} \right\} = \emptyset$, we must have $U = \Omega$. In this case,

$$0 = \kappa_s(x) - \eta_0 e^{Au(x)}, \quad \forall x \in \Omega.$$  

By the continuity,

$$0 = \kappa_s(x) - \eta_0 e^{Au(x)}, \quad \forall x \in \bar{\Omega}. \quad \text{(4.25)}$$
In particular,
\[ 0 = k_s(z) - \eta_0 e^{A}, \quad \forall z \in \partial \Omega_1, \]
\[ 0 = k_s(z) - \eta_0, \quad \forall z \in \partial \Omega_0. \]
So \( \eta_0 = e^{-A} k^1 = k^0 \). We again get
\[ k_s(x) = \frac{e^{A(u(x))-1} k^1}{e^{A u(x)} k^0}. \]
Equation (4.25) implies \( k_s(x) = \text{constant} \) for any \( x \in \Sigma^c \) for all \( 0 \leq c \leq 1 \). In this case, \( \Sigma^c \) must be a round sphere for all \( 0 \leq c \leq 1 \). \( \square \)

**Remark 4.3.** Theorem 1.4 covers all quasilinear equations satisfying the structural conditions (1.8)–(1.10). Therefore, it covers the quasilinear equations treated in [11], [14] from the discussion in Section 2. In particular, \( \varpi \equiv 0 \) if \( F \) is quasilinear. In this case, (1.12) becomes

\[ (4.26) \quad k^c \geq \min \left\{ k^0 e^{4c}; k^1 e^{A(c-1)} \right\}, \quad \forall c \in [0, 1]. \]

From the proof above, the strong maximum principle concludes that if ``='' holds for some \( c_0 \in (0, 1) \) in (4.26), then \( k_s(x) \equiv \text{constant} \) for all \( x \in \Sigma^c \) and for all \( c \in (0, 1) \). This implies that \( \Sigma_{u(x)}^{(3)} \) is a round sphere for every \( x \in \overline{\Omega} \). The same conclusion is also true if condition (1.6) holds. Note that \( \varpi \) was used only in (4.18) to get (4.20). It is proved in [2] that \( S_j \equiv 0 \) under condition (1.6). In that case, one may take \( \varpi = 0 \) in (4.18).

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**References**


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