Abstract—The controllability and observability of leader-following multi-agent linear systems under switching topology are considered. Unlike the existing results, the controllability and observability are analyzed in a different context. More specifically, as for the controllability problem, the admissible control input for each follower agent can only use relative and local information from its neighbors and the control objective is the convergence of each follower’s state to that of the leader agent; as for the observability problem, the output of the multi-agent systems is all the information transmitted in the multi-agent network. It turns out that under the controllability and observability of individual system, the jointly connected switching topology, including fixed topology as a special case, implies the controllability and observability of the multi-agent systems. As applications, these properties are used in the leader-following consensus problem under switching topology.

Index Terms—Multi-agent systems. Controllability, Observability, Joint Connectivity. Leader-following consensus.

I. INTRODUCTION

Multi-agent systems is an area that is currently receiving a lot of interest in a variety of research communities, including biology, robotics, sensor networks, artificial intelligence, automatic control and so on. Network plays a key role of coupling the dynamics of each individual agent, and the topology of the network can be represented by a graph. A central issue arising in the multi-agent systems is to develop distributed control scheme based on local information that enables the agent to achieve collective behaviors, such as consensus, formation, flocking and swarming, etc; for details, see survey papers [10], [11] and references therein.

A relatively new line of research in this area concerns the development of distributed control of high-order multi-agent systems via observers or dynamic output feedback. Recently, the observer-based consensus problem has been investigated in several papers; for example, results for consensus of multiple linear systems via dynamic output feedback are obtained by using low gain approach, $H_{\infty}$ approach and nonlinear matrix inequality approach, presented in [15], [3] and [20], respectively. Some other kinds of dynamic controllers have also been suggested, see for example [19], [4], [8], [5], [21]. However, all the results mentioned here are only suitable for fixed networked topology, being not valid for time-varying topology. This note presents an observer-based control scheme, working well for both fixed and switching topologies.

Obviously, the foundation for the above design is the controllability and observability of the multi-agent system. Considering this, the controllability and observability of leader-following multi-agent linear systems under switching topology are considered. As for the controllability problem, the admissible control input for each follower agent can only use relative and local information from its neighbors and the control objective is the convergence of each follower’s state to that of the leader agent. As for the observability problem, the output of the multi-agent systems is all the information transmitted in the multi-agent network.

Although the controllability and observability problems for multi-agent systems have well addressed in the last decade and a lot of works have been reported, we clarify the contribution of our paper as two-fold: On the one hand, we address the controllability and observability problems for multi-agent systems in a totally different perspective from those in [6], since the input and output of the multi-agent systems considered in this paper are not the same as [6]. For example, the Ref. [6] considers the input and output of a single agent as the input and output of the whole multi-agent systems, and Ref. [17] considers an acting leader as the input of the whole multi-agent systems. One the other hand, our controllability and observability analysis is valid for switching network topology, not only for fixed topology as existing results show.

It turns out that under the controllability and observability of individual system, the jointly connected switching topology, including fixed topology as a special case, implies the
controllability and observability of the multi-agent systems. These properties are used in the observer-based (state-based case is not discussed here due to space limitation) leader-following consensus problem under switching topology.

In our consensus scheme, an observer cohering to each agent is designed. The topology of these observers maintain the same interaction structure as the agent network. Each observer update its state by using its neighbor’s information, including its neighboring observers’ state and its neighboring agents’ output, to give an estimate of the a agent’s state. Thus each observer’s state includes only local information, and consequently the state of observer $i$ being feedback to agent $i$ means that only node $i$’s local information is feedback to agent $i$. This consists our observer-based consensus scheme, which is obviously distributive. Our results show that we can consider the problem of designing an asymptotically stable distributed observer and an asymptotically stable observer-state-feedback control law separately.

The rest of this paper is organized as follows. Section 2 contains the problem formulation and some preliminary results. Section 3 provides observer-based protocols for leader-following consensus over fixed topology. Section 4 extends the result in Section 3 section to switching topology cases. Section 5 presents two illustrated examples. Section 6 is a brief conclusion.

II. SUPPORTING LEMMAS

Four lemmas are provided here for what follows. The proofs of the first three are straightforward, and thus are omitted.

Lemma 1: For any matrices $P, Q_1, \cdots, Q_n$ of appropriate dimensions, the following property holds:

$$\text{rank}(P \otimes \left( \begin{array}{c} Q_1 \\ Q_2 \\ \vdots \\ Q_n \end{array} \right)) = \text{rank}(\left( \begin{array}{c} P \otimes Q_1 \\ P \otimes Q_1 \\ \vdots \\ P \otimes Q_n \end{array} \right))$$

Lemma 2: The system $\dot{x} = Ax + y = Cx$ is observable, if and only if for any matrix $K$ of appropriate dimension, the system $\dot{x} = (A + KC)x$ is observable. Prove:

Lemma 3: Consider an $n$-order differential system $\dot{x}(t) = A_1 x(t) + A_2 y(t)$ with $A_1 \in \mathbb{R}^{n \times n}, A_2 \in \mathbb{R}^{n \times m}$, and $y(t) \in \mathbb{R}^m$. If $A_1$ is Hurwitz and $\lim_{t \to \infty} y(t) = 0$, then $\lim_{t \to \infty} x(t) = 0$.

Lemma 4: [1] Let matrices $H_{i_1}, \cdots, H_{i_m}$ be associated with the graphs $\bar{G}_{i_1}, \cdots, \bar{G}_{i_m}$, respectively. If these graphs are jointly connected, then $\sum_{j=1}^{m} H_j$ is positive definite.

III. CONTROLLABILITY OF MULTI-AGENT SYSTEMS

Consider a multi-agent system consisting of $N$ agents

$$\dot{x}_i = Ax_i + Bu_i \quad (1)$$

and a leader

$$\dot{x}_0 = Ax_0, \quad (2)$$

where $x_0 \in \mathbb{R}^n$ is the state of the leader and $x_i \in \mathbb{R}^n$ is the state of agent $i$, $u_i \in \mathbb{R}^m$ is agent $i$’s input through which the interaction or coupling between agent $i$ and other agents is realized. The matrix $B$ is of full column rank. It is assumed that

Assumption 1: The pair $(A,B)$ is controllable.

We use a graph $G = (\mathcal{V}, \mathcal{E})$ to describe the interaction topology of follower agents, where $\mathcal{V} = \{1, 2, \cdots, N\}$ is the set of nodes representing $N$ agents and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges $(i, j)$, meaning that agents $i$ and $j$ have information exchange. The graph $G$ can be characterized by its Laplacian matrix $L$, see e.g. [11].

The interaction topology of leader and follower agents $\{0, 1, \cdots, N\}$ can be described by an extended graph $\bar{G}$, which consists of graph $G$, node 0 and edges among node 0 and its information-sending follower node $i$. Similarly, the graph $\bar{G}$ can be characterized by its structure matrix $H = L + D$, where $D = \text{diag}(d_1, \cdots, d_N)$ with $d_i = 1$ if node $i$ is a neighbor of node 0, and with $d_i = 0$.

The graph can be time-varying. To describe time dependence of graph, one can use a switching law $\sigma : [0, \infty) \rightarrow \mathcal{P}$, which is a right continuous and piece-wise constant mapping, to describe how the topology switches among a set of graphs indexed by $\mathcal{P}$. That is, the graph at time $t$ can be denoted by $G_{\sigma(t)}$ and $\bar{G}_{\sigma(t)}$, with their structure matrices denoted by $L_{\sigma(t)}$ and $H_{\sigma(t)}$, respectively. It is assumed that there are finite switches in any bounded time interval. Let $N_i(t)$ be the neighbor set of agent $i$ with respect to graph $G_{\sigma(t)}$ at time $t$. The coupling between agent $i$ and its neighbor agents $j \in N_i(t)$ and the leader is realized through the control input $u_j$, which is a feedback or a dynamic feedback of the following local information

$$z_i = \sum_{j \in N_i(t)} (x_j - x_i) + d_i(t)(x_0 - x_i), \quad i = 1, \cdots, N. \quad (3)$$

If there exists, for each agent $i \in \{1, \cdots, N\}$, above control law $u_i$, such that the resulting closed-loop systems (2)-(1) satisfy

$$\lim_{t \to \infty} \|x_i(t) - x_0(t)\| = 0, \quad i = 1, \cdots, N,$$

then the multi-agent systems (2)-(1) are said to achieve leader-following consensus.

Denoting $z_i = x_i - x_0$ for $i = 1, \cdots, N$, then the leader-following consensus problem was transformed into the following collective output feedback problem

$$\dot{\varepsilon}_i = A\varepsilon_i + Bu_i,$$

$$z_i = \sum_{j \in N_i} (\varepsilon_j - \varepsilon_i) + d_i\varepsilon_i,$$

$$u_i = Kz_i, \quad i = 1, \cdots, N. \quad (4)$$

Expressing the error dynamics in a compact form with $\varepsilon = (\varepsilon_1^T, \varepsilon_2^T, \cdots, \varepsilon_N^T)^T$, $u = (u_1^T, u_2^T, \cdots, u_N^T)^T$, one has

$$\left\{ \begin{array}{l}
\dot{\varepsilon}(t) = (I_N \otimes A)\varepsilon(t) + (I_N \otimes B)u(t) \\
z(t) = -(H_{\sigma(t)} \otimes I_n)\varepsilon(t).
\end{array} \right. \quad (5)$$
Therefore, the controllability of multi-agent systems (5) with respect to jointly connected switching signal set \( \mathcal{J}_C \), which will be defined later, is formulated as follows. To this end, denote by \( \phi(t, t_0, x, u, \sigma) \) the trajectory of the multi-agent system (5) with the initial state \( \varepsilon \) at time \( t_0 \), under the switching law \( \sigma \) and the control input \( u \).

**Definition 1:** A state \( \varepsilon \) of multi-agent systems (5) is controllable, if there exist a time instant \( t_f > 0 \) and an input \( u \) which is a functions of \( z \), and a switching signal \( \sigma \in \mathcal{J}_C \), such that \( \phi(t_f, t_0, \varepsilon, u, \sigma) = 0 \). The multi-agent system (5) is said to be controllable if any state is controllable.

Since an essential future of consensus problem is that zero phenomena dose not happen. During each time interval \( [t, t_j+1] \), \( j = 0, 1, \ldots \), the admissible control is the function of \( x \), and the control input can only involve relative and distributed feedback of \( z \), such that \( \phi(t_j, t_0, \varepsilon, u, \sigma) = 0 \). The multi-agent system (5) is said to be controllable if any state is controllable.

**Assumption 2:** The graphs are jointly connected across each interval \( [t_k, t_{k+1}] \), \( k = 0, 1, \ldots \)

Note that the multi-agent system (6) is a switched system, and the set of controllable state of a switched system forms a subspace [16]. Define by \( C \) the controllable subspace of multi-agent system (6). Similar to the controllability analysis of switched linear system in [16, pp. 120-122], the expression of \( C \) can be obtained as

\[
C = \sum_{k=0}^{\infty} \sum_{i=1}^{m_k} (I_N \otimes A)^i Im(H_{k_j} \otimes B).
\]

Noting that all possible graphs are finite, the above infinite sum has essentially finite terms. Define, for \( k = 1, 2, \ldots \),

\[
C_{(t_k,t_{k+1})} = (H_{k_1} \otimes \mathcal{E}, H_{k_2} \otimes \mathcal{E}, \ldots, H_{k_{m_k}} \otimes \mathcal{E}),
\]

where \( \mathcal{E} = (B, AB, \ldots, A^{n-1}B) \) which is full rank due to Assumption 1. Note tat the multi-agent system (6) is controllable whenever \( C = \mathbb{R}^{nN} \) or \( \text{rank}(C) = nN \) [16]. These latter conditions can be guaranteed if \( \text{rank}(C_{(t_k,t_{k+1})}) = nN \) for all \( k = 0, 1, \ldots \), which can be furthered assured by joint connectivity of the graph, as shown below. Now everything is prepared to formulate the

**Theorem 1:** Under Assumption 1 and Assumption 2, the observability matrices (7) of multi-agent system (5) are full rank, and consequently the multi-agent system (5) is controllable.

**Proof:** It can be seen that

\[
\text{rank}(C_{(t_k,t_{k+1})}) = \text{rank}((H_{k_1}, \ldots, H_{mk}) \otimes \mathcal{E}) = \text{rank}((H_{k_1}, \ldots, H_{mk}) \cdot \text{rank}(\mathcal{E})) = n \cdot \text{rank}((H_{k_1} + \cdots + H_{mk}, \ldots, H_{mk})).
\]

Since, for any \( [t_k, t_{k+1}] \), the graph is jointly connected across it, then the matrix \( H_{k_1} + \cdots + H_{mk} \) is positively definite, referring to Lemma 4. Thus, \( \text{rank}((H_{k_1} + \cdots + H_{mk}, \ldots, H_{mk})) = N \). Therefore, \( \text{rank}(C_{(t_k,t_{k+1})}) = nN \) for \( k = 0, 1, \ldots \), and consequently the multi-agent system (5) is controllable.

**IV. OBSERVABILITY OF MULTI-AGENT SYSTEMS**

In this section we assume that the full state of each agent is unavailable. Now the multi-agent system we consider consists of a leader

\[
\begin{align*}
\dot{x}_0 &= Ax_0, \\
y_0 &= Cx_0
\end{align*}
\]

and \( N \) follower agents

\[
\begin{align*}
\dot{x}_i &= Ax_i + Bu_i, \\
y_i &= Cx_i,
\end{align*}
\]

where \( y_0 \) and \( y_i \) are the outputs of the leader and follower agent \( i \), respectively. The following standard assumption is made in this section.

**Assumption 3:** The pair (A,C) is observable.
In this case, the local information in (3) is modified as (use the same notation $z_i$ for simplicity)

$$z_i = \sum_{j \in N_i(t)} (Cx_j - Cx_i) + d_i(t)(Cx_0 - Cx_i), \quad (10)$$

with $i = 1, \ldots, N$. Then similar to (5), one has

$$\begin{cases} \dot{\varepsilon}(t) = (I_N \otimes A)\varepsilon(t) + (I_N \otimes B)u(t) \\ z(t) = -(H_{\sigma(t)} \otimes C)\varepsilon(t). \end{cases} \quad (11)$$

This section investigates the observability of the multi-agent systems (11), where the outputs are the information transmitted in the network.

**Definition 3:** A state $\varepsilon$ of multi-agent systems (11) is said to be unobservable, if for any jointly connected switching signal $\sigma \in \mathcal{J}_C$, it has $(H_{\sigma} \otimes C)\phi(t;0,\varepsilon,\sigma) = (H_{\sigma} \otimes C)\phi(t;0,\sigma), \forall t > 0$. The unobservable set of (11) is the set of states which are unobservable. The system (11) is said to be observable if its unobservable set is null.

Note that $\cdots < t^1_k < \cdots < t^m_k < t^1_{k+1} < \cdots < t^m_{k+1} < \cdots$ are the switching instants of the switching signal $\sigma$. Arguing in a manner similar to the observability analysis of switched systems in [16, pp.131], it can be follows from Definition 3 that for an unobservable state $\varepsilon$ of multi-agent systems (11), and for each $k = 1, 2, \cdots$, one has

$$(H_k \otimes C)(I_N \otimes A)\varepsilon = 0,$$

for all $k \in \{\sigma(t^l_k)|j = 1, 2, \cdots, m_k\}$ and for all $l = 1, \cdots, n - 1$. Defining, for $k = 1, 2, \cdots$,

$$\mathcal{O}_{t_k,t_{k+1}} = \begin{pmatrix} H_{k_1} \otimes C \\ \vdots \\ H_{k_1} \otimes C^{n-1} \\ H_{k_m} \otimes C \\ \vdots \\ H_{k_m} \otimes C^{n-1} \end{pmatrix} \quad (12)$$

one has $\varepsilon \in \ker(\mathcal{O}_{t_k,t_{k+1}})$. Therefore, according the definition of observability above, the multi-agent system is observable if the observability matrix (12) has full rank for each $k = 1, 2, \cdots$.

**Theorem 2:** Under Assumption 3 and Assumption 2, the observability matrices (12) of multi-agent system are full rank, and consequently the multi-agent systems (11) are observable.

Proof: It can be seen that

$$\text{rank}(\mathcal{O}_{t_k,t_{k+1}}) = \text{rank} \begin{pmatrix} H_{k_1} \otimes C \\ \vdots \\ H_{k_1} \otimes C^{n-1} \\ H_{k_m} \otimes C \\ \vdots \\ H_{k_m} \otimes C^{n-1} \end{pmatrix} = \text{rank} \begin{pmatrix} H_{k_1} \otimes C \\ \vdots \\ H_{k_m} \otimes C \\ \vdots \\ H_{k_m} \otimes C^{n-1} \end{pmatrix}$$

By Lemma 1, one has

$$\text{rank}(\mathcal{O}_{t_k,t_{k+1}}) = \text{rank} \begin{pmatrix} H_{k_1} \\ \vdots \\ H_{k_m} \end{pmatrix} \otimes \begin{pmatrix} C \\ \vdots \\ C^{n-1} \end{pmatrix}$$

Therefore, by the property that the rank of the Kronecker product of two matrices is equal to the product of the two ranks of both matrices, one has

$$\text{rank}(\mathcal{O}_{t_k,t_{k+1}}) = \text{rank} \begin{pmatrix} H_{k_1} \\ \vdots \\ H_{k_m} \end{pmatrix} \otimes \begin{pmatrix} C \\ \vdots \\ C^{n-1} \end{pmatrix} = n \cdot \text{rank} \begin{pmatrix} H_{k_1} \\ \vdots \\ H_{k_m} \end{pmatrix}$$

Since, for any $[t_k,t_{k+1})$, the graph is jointly connected across it, then the matrix $H_{k_1} + \cdots + H_{k_m}$ is positively definite, referring to Lemma 4. Thus,

$$\text{rank} \begin{pmatrix} H_{k_1} \\ \vdots \\ H_{k_m} \end{pmatrix} = \text{rank} \begin{pmatrix} H_{k_1} + \cdots + H_{k_m} \\ \vdots \\ H_{k_m} \end{pmatrix} = N,$$

which completes the proof. \hfill \blacksquare

**Remark 1:** It follows from Lemma 2 and Theorem 2 that the following system

$$\begin{cases} \dot{\varepsilon}(t) = (I_N \otimes A - H_{\sigma(t)} \otimes GC)\varepsilon(t) \\ z(t) = -(H_{\sigma(t)} \otimes C)\varepsilon(t) \end{cases} \quad (13)$$

is observable.
V. LEADER-FOLLOWING CONSENSUS OF MULTI-AGENT SYSTEMS

This section is devoted to leader-following consensus of multi-agent system (11) using observer-based control law. The following assumption is furtherly made in this section.

Assumption 4: The matrix $A$ is neutrally stable. This assumption is equivalent to the fact that there exists a positive definite matrix $P$ such that

$$PA + ATP \leq 0,$$

(15)

or to the fact that the motion of leader is in a bounded region. This assumption is widely used in consensus of multiple linear systems, for example [3], [13], [15], [18].

The following observer-based feedback controller for each agent $i = 1, \cdots, N$ is proposed:

$$\hat{\xi}_i = A \hat{\xi}_i + G(\hat{z}_i - z_i) + Bu_i, u_i = K \hat{z}_i,$$

(16)

where

$$\hat{z}_i = \sum_{j \in N(t)} (C \hat{\xi}_j - C \hat{\xi}_i) + d_i(t)C \hat{\xi}_i,$$

(17)

here the matrix $K$ is chosen such that $A + BK$ is Hurwitz, and $G$ is designed as follows:

- Find a positively definite matrix $P$ satisfying (15);
- The matrix $G$ is obtained as

$$G = P^{-1}C^T.$$

(18)

Since $z_i$ and $\hat{z}_i$ are local, the observer is essentially distributed, thus feeding the state of each observer back to the corresponding agent is again a distributed control scheme. Then we have the following

Theorem 3: Consider the multi-agent systems (8)-(9) connected by switching graphs, under the Assumptions 1-4. Then, with the matrices $G$ and $K$ given above, the dynamic controls (16)-(17) solve the leader-following consensus problem.

Proof: The systems (8)-(9) under the observer-based controller (16)-(17) result in

$$\ddot{\xi}(t) = (IN \otimes (A + BK))\dot{\xi}(t) + (IN \otimes BK)\xi(t),$$

$$\dot{\xi}(t) = (IN \otimes A - H_{\sigma(t)} \otimes GC)\xi(t).$$

(19)

Obviously, the second one is a switched system.

First, we prove $\lim_{t \to \infty} \dot{e}(t) = 0$. To this end, considering a candidate Lyapunov function $V(t) = \dot{e}(t)(IN \otimes P)\dot{e}(t), P > 0$, and arguing in a manner similar to the proof of [9, Theorem 2], one obtains $\lim_{t \to \infty} CE_i(t) = 0, i = 1, \cdots, N$. Therefore, $\dot{z}(t) = [H_{\sigma(t)} \otimes (GC)]\dot{e}(t) \to 0$ as $t$ goes to infinity. On the other hand, under Assumptions 1-4 and by referring to Theorem 2 and Remark 1, it has that the system (13) is observable. Thus $\lim_{t \to \infty} \dot{e}(t) = 0$ implies $\dot{e}(t) = 0$.

Since $\lim_{t \to \infty} \dot{e}(t) = 0$ and $IN \otimes (A + BK)$ is Hurwitz, then from (19) and by using Lemma 3 that $\lim_{t \to \infty} \dot{e}(t) = 0$. This completes the proof.

VI. SIMULATIONS

Consider a multi-agent system consisting of a leader and four agents. Assume the system matrices are

$$A = \begin{pmatrix} -1.0159 & 0.6217 & 1.1382 \\ -1.2048 & 0.0518 & 1.3922 \\ 3.3052 & 0.5121 & -4.0359 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, C = \begin{pmatrix} 3 & 3 & -8 \\ 3 & 5 & -2 \end{pmatrix}.$$  

The eigenvalues of $A$ are $\{-5, i, -i\}$. We suppose that possible interaction graphs are $\{G_1, G_2, G_3, G_4, G_5, G_6\}$ which are as shown in [9, Figure 5(a)] and the interaction graphs are switching as $G_1 \to G_2 \to G_3 \to G_4 \to G_5 \to G_6 \to G_1 \to \cdots$, with graph being active for 1 second. Since the graphs $G_1 \cup G_2 \cup G_3$ and $G_4 \cup G_5 \cup G_6$ are connected, we can choose $t_k = k, t_{k+1} = k + 3$ and $t_k^1 = k + 1, t_k^2 = k + 2, t_k^3 = k + 3$ with $k = 0, 1, \cdots$. Using Matlab, the matrix $K$ such that $A + BK$ is Hurwitz and the matrix $G$ satisfying (18) can be obtained as

$$K = \begin{pmatrix} 6.9157 & -2.3898 & 0.7023 \\ -5.5953 & 1.3432 & -0.7400 \end{pmatrix},$$

$$G = \begin{pmatrix} 4.1200 & 3.5200 \\ -25.0000 & -18.4600 \end{pmatrix}.$$  

The trajectories of $\epsilon_i = x_i - x_0, i = 1, 2, 3, 4$ are show in Figure 1. We can see that all the agents follow the leader.

VII. CONCLUSIONS

The controllability and observability of leader-following multi-agent linear systems under switching topology are considered. The admissible control for the controllability involves only relative and local information from its neighbors and the control objective is the convergence of each follower’s state to that of the leader agent. As for the observability problem, the output of the multi-agent systems is all the information transmitted in the multi-agent network. It turns out that under the controllability and observability of individual system, the jointly connected switching topology, including fixed topology as a special case, implies the controllability and observability of the multi-agent systems. These properties are used in the leader-following consensus problem under switching topology.

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The error trajectories between the leader and four agent in switching topology.


