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Long term evolution of Molniya orbit under the effect of Earth’s non-spherical gravitational perturbation

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Abstract

A double resonance model is applied to study the long term evolution of a Molniya orbit, which is highly elliptical ($e \geq 0.7$), critically inclined ($i \approx 63.4^\circ$), and in the state of the 2:1 mean motion resonance with the Earth rotation. The dynamics of a Molniya orbit can be divided into three kinds: short (12 hours), intermediate (several years) and long (several centuries) period motions, with the latter two studied in this paper. The $J_2$ and $J_{l2}$ ($l = 2, 3, \ldots, 8$) harmonics are modelled, based on a careful selection. The analytic solution for the intermediate period motion is obtained, a first integral, $\bar{I}_3$, for the long period motion is derived analytically, and the phase structures are obtained by the level curves of $\bar{I}_3$. Three types of the phase structures, depending on the equilibria and stabilities, are observed when the Hamiltonian constant varies. Compared with the near circular 12-hour satellite orbits and with the critically inclined orbits without mean motion resonance with the Earth rotation, the features of the Molniya orbits are discussed in detail. It is pointed out that (1) unlike the case of near circular orbits, the $J_{32}$ term does not dominate the 2:1 mean motion resonance prob-
lem (intermediate period motion), and that (2) instead of the $J_2^2$ terms, the resonant tesseral harmonics dominate the critical inclination problem (long period motion).

**Keywords:** Molniya orbit, mean motion resonance, critical inclination, double resonance, large eccentricity

### 1. Introduction

On the basis of the glorious products of the orbit theories, various special artificial satellite orbits are developed for certain missions, one of which is the Molniya orbit. It is a type of highly elliptical orbit with an inclination of $63.4^\circ$, an argument of perigee of $270^\circ$ and an orbital period of one half of a sidereal day. The dynamics of a Molniya orbit is complicated due to the combined effects of the critical inclination and the 2:1 mean motion resonance. The large eccentricity, usually over 0.7, adds some difficulties to the problem.

The mean motion resonance and the critical inclination problem have been intensively studied since 1960s. The principal perturbations on the near circular geosynchronous and semi-synchronous satellites orbiting the Earth arise from the $J_{22}$ and $J_{32}$ tesseral harmonics, respectively (Blitzer et al., 1962; Blitzer, 1965; Gedeon, 1969; Nacozy and Diehl, 1982; Sochilina, 1982; Sampaio et al., 2012; Zhao et al., 2013; Zhang et al., 2013). The well known Von Zeipel correction terms, proportional to $J_2^2$ in the averaged Hamiltonian, dominate the classical critical inclination problem (Hori, 1960; Jupp, 1975, 1980, 1987). The Ideal Resonance Model, firstly presented by Garfinkel, is applied to study both problems (Garfinkel, 1966; Deprit, 1969; Jupp, 1972, 1973; Garfinkel and Williams, 1974; Garfinkel, 1975). Using this model, Liu et al. (1991) derives the libration width and the center period, considering the $J_{22}$ ($J_{32}$) tesseral terms for the 24-hour (12-hour) orbits, and the $J_2^2$ terms in the classical critical inclination problem. They present that the libration width of $i$ is $\Delta i \approx 0.04^\circ$ for the near circular, critically inclined low Earth orbits. This width is so small that the phase structure of the resonance is easy to disrupt under the effects of other perturbations.

However, the combined effects of the critical inclination and the mean motion resonance have rarely been investigated. This problem can be reduced to a double resonance model. Henrard (1990) presents a semi-numerical perturbation method for separable two-degree-of-freedom Hamiltonian systems,
and his main idea is that to numerically integrate the motion of the separated degree-of-freedom and then to obtain a first integral, independent of the Hamiltonian. Delhaise and Henrard (1993) reduce the combined resonance problem to a near separable double resonance model, in view of the existence of two kinds of oscillations with one’s period approximately 100 times larger than the other’s. Hence, they apply the semi-numerical method to this problem to obtain the global secular dynamics.

Henrard’s semi-numerical method is generally applicable to the most separable or near separable two-degree-of-freedom Hamiltonian systems. Instead, analytic methods are discussed and applied in this paper to analyse the long term evolution of the Molniya orbits. That is reasonable because the intermediate period motion of such orbits can be well approximated by the pendulum model. A detailed description of the study logic is provided in the following.

In section 2, a simplified double resonance model (Eq. 10) is established to describe the long term evolution of the Molniya orbits. Equilibria and their stabilities are studied, and the results are listed in Table 1. The double resonance model is separated into two one-degree-of-freedom Hamiltonian systems (Eq. 16), according to the magnitude of the coefficients in Eq. (10).

In section 3, the unperturbed part of the Hamiltonian system, determined by $H_0$, is solved. The solution listed in Eq. (19)–Eq. (22) is a good approximation to the intermediate period motion of the Molniya orbits.

In section 4, the long period motion of the Molniya orbits is studied qualitatively. The long term perigee evolution of the Molniya orbits are compared with that of the critically inclined orbits without mean motion resonance. The $J_2^2$ terms dominate the long term perigee motion if the orbit is critically inclined and is not in the state of mean motion resonance. In the combined resonance problem (i.e. the Molniya orbits), the resonant tesseral terms dominate, since they cannot be removed as short periodic terms. The libration widths of the two problems, presented in Eqs. (27) and (30), are compared to show that the mean motion resonance enhances the perigee resonance (i.e. the so-called critical inclination problem).

In section 5, the phase structures of the long period motion are studied, based on the analytic expression of a first integral $I_3$. Depending on the fact that the intermediate period motion librates or circulates, three cases (A, B and C) are discussed separately. Three basic types (I, II, III), depending on the equilibria and their stabilities, of the phase structures are discovered by choosing different values of the Hamiltonian constant $h$. All of these three
types are observed in case A and only one type (II) is observed in cases B and C. Anomalies are detected in case C when the intermediate period motion is near the separatrix between libration and circulation. Numerical tests are applied to verify the phase structures of the long period motion at the end of the section 5.

2. Double resonance model

2.1. Construction of the model

Since a Molniya orbit is in the state of the 2:1 mean motion resonance, we introduce the following canonical variables (Giacaglia, 1969; Delhaise and Henrard, 1993):

\[ x_1 = \ell, \quad x_2 = g, \quad x_3 = \frac{1}{2}\ell + h \]

\[ y_1 = L - \frac{1}{2}H, \quad y_2 = G, \quad y_3 = H \]

with \(L, G, H, \ell, g, h\) being the Delaunay variables:

\[ L = \sqrt{\mu a}, \quad G = L\sqrt{1 - e^2}, \quad H = G\cos i \]
\[ \ell = M, \quad g = \omega, \quad h = \Omega - S_g \]

where \(a, e, i, M, \omega, \Omega\) are the keplerian elements and \(S_g\) is the sidereal time.

The Hamiltonian for a Molniya orbit is averaged over the short period motion, considering the Earth’s \(J_2\) harmonic and the tesseral harmonics. Up to the second order of the \(J_2\) harmonics, it takes the following form (Delhaise and Henrard, 1993):

\[ \mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1 + \mathcal{F}_{2T} + \mathcal{F}_{2Z} \]

with

\[
\begin{align*}
\mathcal{F}_0 &= -\frac{\mu^2}{2L^2} - \omega \epsilon H \\
\mathcal{F}_1 &= -\frac{\mu^4 R_e^2 J_2}{L^5} \left( 3 \frac{H^2}{G^2} - 1 \right) \left( \frac{L}{G} \right)^3 \\
\mathcal{F}_{2T} &= -\sum_{m=2(l-2p+q)} B_l m p q(a, e, i) S_{l m p q}(x_2, x_3) \\
\mathcal{F}_{2Z} &= -\frac{\mu^6 R_e^2 J_2^2}{4L^10} \left[ A(L, G, H) \cos 2x_2 + C(L, G, H) \right]
\end{align*}
\]
where the function $S_{lmp}$ takes the form of

$$S_{lmp}(x_2, x_3) = \frac{\cos \left[ m(x_3 - \lambda_{lm}) + (l - 2p)x_2 \right]}{\sin l} \quad l - m \text{ even}$$

$$S_{lmp}(x_2, x_3) = \frac{\sin \left[ m(x_3 - \lambda_{lm}) + (l - 2p)x_2 \right]}{\cos l} \quad l - m \text{ odd}$$

(5)

In Eq. (4), $A, C$ and $B_{lmpq}$ are functions of $L, G, H$ or $a, e, i$, and their expressions are listed in the Appendices (see Eqs. A.1–A.3) for convenience. $\mu$ is the product of the Newtonian constant of gravitation by the mass of the Earth, $\omega_e$ is the Earth rotation angular velocity, $R_e$ is the mean equator radius of the Earth and $\lambda_{lm}$ is the longitude of the major axis of symmetry of the $(l, m)$ spherical harmonic.

We first simplify the Hamiltonian by removing the negligible terms. $B_{lmpq}$, $\frac{J_2^2}{4L_{10}}A$ and $\frac{J_2^2}{4L_{10}}C$ are evaluated for a Molniya orbit, i.e. let $a \approx 26,560$ km, $e \approx 0.72$ and $i \approx 63.4^\circ$, and the calculated values are:

$$B_{2,2,0,-1} \approx 8.1 \times 10^{-8}, \quad B_{3,2,2,2} \approx -1.06 \times 10^{-8}$$

$$B_{3,2,1,0} \approx -5.8158 \times 10^{-9}, \quad B_{3,2,0,-1} \approx -1.17064 \times 10^{-9}$$

$$\ldots$$

$$\frac{J_2^2}{4L_{10}}A \approx -4.8 \times 10^{-10}, \quad \frac{J_2^2}{4L_{10}}C \approx -1.1 \times 10^{-10}$$

(6)

Hence it is reasonable to disregard the $J_2^2$ terms as a whole. In addition, the $J_{32}$ term is slightly smaller than the $J_{22}$ term, and they (the $J_{22}$ and $J_{32}$ terms) are larger than the other degree and order tesseral terms, and hence $F_{2T}$ can be reformulated as

$$F_{2T} = \sum_{m \text{ even}} (A_m(x_2; a, e, i, \lambda_{lm}) \cos mx_3 + B_m(x_2; a, e, i, \lambda_{lm}) \sin mx_3)$$

where the expressions of $A_m$ and $B_m$ take the following form:

$$A_m = -\sum_{l \text{ even}} B_{lmpq}(a, e, i) \cos \eta_{lmp} - \sum_{l \text{ odd}} B_{lmpq}(a, e, i) \sin \eta_{lmp}$$

$$B_m = \sum_{l \text{ even}} B_{lmpq}(a, e, i) \sin \eta_{lmp} - \sum_{l \text{ odd}} B_{lmpq}(a, e, i) \cos \eta_{lmp}$$

(7)

with $\eta_{lmp} = (l - 2p)x_2 - m\lambda_{lm}$. Then the terms with $m \geq 4$ are removed since they are much smaller than the terms with $m = 2$ for a Molniya orbit.
Eventually, the Hamiltonian

$$F = F_0 + F_1 + F_{2T}$$

with \( l = 2, 3, \ldots, 8 \) and \( m = 2 \) in the function \( F_{2T} \) describes the motion of the Molniya orbits, as presented in this paper below.

2.2. Simplified double resonance model

\( F \) determines a two-degree-of-freedom Hamiltonian system since the fast variable \( x_1 \) is removed. The critical values \( y_2^c \) and \( y_3^c \) are determined by

$$\frac{\partial(F_0 + F_1)}{\partial y_2} = 0, \quad \frac{\partial(F_0 + F_1)}{\partial y_3} = 0.$$  \( 9 \)

To study the long term evolution of a Molniya orbit, we restrain the canonical variables \((x_2, x_3, y_2, y_3)\) in some neighbourhoods of the critical values, and it is called the area of interest.

The Hamiltonian \( F \) is expanded with respect to \( y_2 \) and \( y_3 \) at the critical values. Then a simplified double resonance model, determined by the Hamiltonian \( H \), is obtained:

$$H(x_2, x_3, p_2, p_3) = \frac{1}{2} w_{22} p_2^2 + w_{23} p_2 p_3 + \frac{1}{2} w_{33} p_3^2 + A_2(x_2) \cos 2x_3 + B_2(x_2) \sin 2x_3$$

where \( p_i = y_i - y_i^c \) \((i = 2, 3)\) and \( w_{ij} \) \((i, j = 2, 3)\) are the second order partial derivatives of \( F_0 + F_1 \) with respect to \( y_2 \) and \( y_3 \) at the critical values:

$$w_{ij} = \frac{\partial^2(F_0 + F_1)}{\partial y_i \partial y_j} \bigg|_{y_2 = y_2^c, y_3 = y_3^c} , \quad i, j = 2, 3$$

(11)

Considering the periodicity of the function \( F \) with respect to \( x_2 \) and \( x_3 \), and the magnitudes of \( w_{ij} \) and \( F_{2T} \), the area of interest is defined by \( x_2 \in [0, 2\pi] \), \( x_3 \in [0, \pi] \), \(|p_2| < 0.04\) and \(|p_3| < 0.005\). The restriction on the values of \( p_2 \) and \( p_3 \) is chosen that they are a little larger than the libration width. For \( p_3 \), the exact value of libration width is calculated in Eq. (24). For \( p_2 \), the values are obtained through the phase structures (presented in section 5) of the long period motion. Instead of \( \Delta p_2 \), the width of the eccentricity \( \Delta e \) and the inclination \( \Delta i \) are listed to clarify the physical meaning. The linear and
higher order terms in the expansion of $\mathcal{F}_{2T}$ are neglected in this area. Hence $A_2$ and $B_2$ are one variable functions of $x_2$.

Give the constant value of $y_1$ resulting in this set:

$$a^c = 26,554.3 \text{ km, } e^c = 0.7227, \ i^c = 63.4349^\circ \quad (12)$$

Correspondingly, the values of $w_{ij}$ are

$$w_{22} = -3.4273 \times 10^{-5}, \quad w_{23} = 7.66367 \times 10^{-5}, \quad w_{33} = -0.0432686 \quad (13)$$

and $A_2, B_2$ are of magnitude of $10^{-8}$ for all $x_2$, where the normalized unit system is applied, i.e.

$$[L] = R_e, \quad [M] = \text{Mass of Earth } M_e, \quad [T] = \sqrt{R_e^3/\mu} \approx 806.81038 \quad (14)$$

and in this unit system, $\mu = 1$.

2.3. Equilibria and stabilities

Eq. (10) determines a simplified double resonance model. Let the equilibrium be $(x_2^*, x_3^*, p_2^*, p_3^*)$. Then it is obvious that $p_2^* = p_3^* = 0$, and $x_2^*, x_3^*$ are determined by

$$\begin{cases}
0 = \frac{\partial \mathcal{F}_{2T}^c}{\partial x_2} = A_2'(x_2) \cos 2x_3 + B_2'(x_2) \sin 2x_3 \\
0 = \frac{\partial \mathcal{F}_{2T}^c}{\partial x_3} = -2A_2(x_2) \sin 2x_3 + 2B_2(x_2) \cos 2x_3 
\end{cases} \quad (15)$$

where $\mathcal{F}_{2T}^c(x_2, x_3) = \mathcal{F}_{2T}(x_2, x_3, y_2^c, y_3^c)$.

According to the theory of the Hamiltonian dynamics, if $(x_2^*, x_3^*, p_2^*, p_3^*)$ is a extremal point of the Hamiltonian $\mathcal{F}$, then it is a stable equilibrium, if $(x_2^*, x_3^*, p_2^*, p_3^*)$ is a saddle point, then it is an unstable equilibrium. In this problem, $(p_2^*, p_3^*) = (0, 0)$ is a local maximum point of $\mathcal{F}_0 + \mathcal{F}_1$, seen as a two variable function of $p_2$ and $p_3$. Hence, if $(x_2^*, x_3^*)$ is a local maximum (minimum) point of $\mathcal{F}_{2T}^c$ as a two variable function of $x_2$ and $x_3$, then $(x_2^*, x_3^*, p_2^* = 0, p_3^* = 0)$ is a stable (unstable) equilibrium of the Hamiltonian system determined by $\mathcal{F}$. Saddle points, $(x_2^*, x_3^*)$, of $\mathcal{F}_{2T}^c$ correspond to unstable equilibria. The Newton iteration algorithm is applied to solve Eq.(15) and the results are listed in Table 1, and saddle points of $\mathcal{F}_{2T}^c$ are not listed here.
Table 1: Equilibria and their stabilities.*

<table>
<thead>
<tr>
<th>$x_2(^\circ)$</th>
<th>$x_3(^\circ)$</th>
<th>stability</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.7604</td>
<td>70.25</td>
<td>unstable</td>
</tr>
<tr>
<td>86.743</td>
<td>74.34</td>
<td>stable</td>
</tr>
<tr>
<td>173.64</td>
<td>78.06</td>
<td>unstable</td>
</tr>
<tr>
<td>266.33</td>
<td>75.65</td>
<td>stable</td>
</tr>
</tbody>
</table>

* Extremal points of $F_{xy}$ as a function of $x_2$ and $x_3$ are listed. Here the local maximum points correspond to the stable equilibria. The local minimum points correspond to the unstable equilibria. The Keplerian elements $a, e, i$ in the coefficients $B_{lmpq}(a, e, i)$ are fixed at the critical values (see Eq. 12).

2.4. Separation to two one-degree-of-freedom systems

In view of the magnitude of $w_{ij}$, the Hamiltonian $\mathcal{H}$ in Eq. (10) is divided into a main part $\mathcal{H}_0$ and a perturbation part $\mathcal{H}_1$ as follows:

$$
\begin{align*}
\mathcal{H}_0 &= \frac{1}{2}w_{33}p_3^2 + A_2(x_2)\cos 2x_3 + B_2(x_2)\sin 2x_3 \\
\mathcal{H}_1 &= \frac{1}{2}w_{22}p_2^2 + w_{23}p_2p_3
\end{align*}
$$

(16)

Actually, the double resonance model is separated into two one-degree-of-freedom systems. Then the motion in the 4 dimensional phase space can be divided into an intermediate period motion and a long period motion. This separation is valid outside the chaotic area, i.e. the small neighbourhoods of the unstable equilibria, in which the period of the ‘intermediate period motion’ is too long to be separated.

3. Intermediate period motion

The intermediate period motion is dominated by the main part $\mathcal{H}_0$, which determines a one-degree-of-freedom Hamiltonian system since $y_2$ is excluded. Any constant $h_0$, as a parameter, determines a solution through $\mathcal{H}_0 = h_0$.

Some auxiliary functions $\alpha(x_2)$, $\theta(x_2)$ and $\kappa_{1,2}(x_2, h_0)$ are defined as fol-
lows:

\[
\begin{align*}
\alpha(x_2) &= \sqrt{A_2^2(x_2) + B_2^2(x_2)} \\
\cos 2\theta(x_2) &= \frac{A_2(x_2)}{\alpha(x_2)}, \quad \sin 2\theta(x_2) = \frac{B_2(x_2)}{\alpha(x_2)}, \\
\kappa_1(x_2, h_0) &= \sqrt{\frac{\alpha(x_2) - h_0}{2\alpha(x_2)}}, \quad \kappa_2(x_2, h_0) = \sqrt{\frac{2\alpha(x_2)}{\alpha(x_2) - h_0}}
\end{align*}
\]

Then \( H_0 \) takes the form of

\[
H_0 = \frac{1}{2} \omega_{33} p_3^2 - 2\alpha \sin^2(x_3 - \theta) + \alpha
\]

note that \( \omega_{33} < 0, \alpha > 0 \) and \( |\alpha/\omega_{33}| \approx 10^{-6} \).

3.1. Solution of the intermediate period motion

It is known that the Hamiltonian \( H_0 \) describes a standard pendulum model, with \( x_3 = \theta + k\pi \) and \( x_3 = \theta + \pi/2 + k\pi \) (\( k \in \mathbb{Z} \)) be the stable and unstable equilibria, respectively.

The global analytic solution of a pendulum is presented in most of the textbooks of theoretical mechanics, we then refer to the results directly (Markeyev, 2006). Under the assumption that the initial values are \( x_0^0, p_0^0, x_0^2 = \theta \), and that \( p_0^3 \) is determined by \( h_0 = H_0 \), we present the solution of the \( H_0 \)-system in the following cases:

1. If \( 0 < \kappa_1 < 1 \) or \( |h_0| < \alpha \), \( x_3 \) librates around its centre \( \theta \). The libration period \( T_3 \) is

\[
T_3 = \frac{2K(\kappa_1)}{\sqrt{-w_{33}\alpha}}
\]

The solution can be expressed as

\[
\begin{align*}
x_3 &= \theta + \arcsin(\kappa_1 \text{sn}(u, \kappa_1)) \\
p_3 &= 2\kappa_1 \sqrt{-\alpha/\omega_{33}} \text{cn}(u, \kappa_1) \\
x_2 &= x_0^2 \\
p_2 &= p_0^2 + \frac{T_3}{4K(\kappa_1)} \left[ 2\kappa_1 \text{S}(x_2)(\text{cn} u - 1) + C_1(x_2) \left( 2E(\varphi, \kappa_1) - F(\varphi, \kappa_1) \right) \right]
\end{align*}
\]

where \( u = -(4K(\kappa_1)/T_3) t \) and \( \varphi = \text{am}(u, \kappa_1) \) is the so-called amplitude of the Jacobi elliptic functions.
2. If $0 < \kappa_2 < 1$ or $h_0 < -\alpha$, $x_3$ circulates or evolves monotonously with the time. The circulation period $T_3$ (i.e. the time for $x_3$ evolves from $x_3^0$ to $x_3^0 \pm 2\pi$) is

$$T_3 = 2\kappa_2 K(\kappa_2) / \sqrt{-w_{33} \alpha}$$

(21)

The solution can be expressed as

$$
\begin{cases}
  x_3 = \theta + \text{am} (u, \kappa_2) \\
  p_3 = \frac{2}{\kappa_2} \sqrt{-\alpha/w_{33}} \text{dn} (u, \kappa_2) \\
  x_2 = x_2^0 \\
  p_2 = p_2(0) + \frac{T_3}{4\kappa_2^2 K(\kappa_2)} \left[ 2S_1(x_2) \text{dn} (u, \kappa_2) + C_1(x_2) \left( 2E(\varphi, \kappa_2) - (2 - \kappa_2^2) F(\varphi, \kappa_2) \right) \right]
\end{cases}
$$

(22)

where $u = -(2\sqrt{-w_{33} \alpha} / \kappa_2)t$ and $\varphi = \text{am} (u, \kappa_2)$

3. $\kappa_1 = \kappa_2 = 1$ or $h_0 = -\alpha$ corresponds to the separatrix and $\kappa_1 = 0$ or $h_0 = \alpha$ corresponds to the stable equilibrium.

4. $h_0 > \alpha$ corresponds to the non-existence of the solution.

In the above expressions, $K$, $E$ and $F$ are the Jacobi elliptic integrals, $\text{sn}$, $\text{cn}$, $\text{dn}$ are the Jacobi elliptic functions with $\text{am}$ being the amplitude. $S_1$ and $C_1$ are functions of $x_2$ defined by

$$S_1(x_2) = A'_2 \sin 2\theta - B'_2 \cos 2\theta, \quad C_1(x_2) = A'_2 \cos 2\theta + B'_2 \sin 2\theta$$

(23)

3.2. Characteristics of the intermediate period motion

The centre period and the libration width are the important values to describe the phase structures of the intermediate period motion. The centre period is the value of $T_3$ when $x_3 = \theta$ (i.e. $\kappa_1 = 0$), the libration width is the value of $p_3$ when $h_0 = \alpha$ and $x_3 = \theta$ (i.e. $u = 0$, $\kappa_1 = 1$). They take the form of

$$T_{3\text{centre}} = \pi / \sqrt{-w_{33} \alpha}, \quad \Delta p_3_{\text{max}} = \sqrt{-4\alpha/w_{33}}$$

(24)

An example is listed in Table 2, in which $x_2$ is fixed at $\pm 90^\circ$. 

10
Table 2: Libration centre, width and centre period of intermediate period motion.**

<table>
<thead>
<tr>
<th></th>
<th>Centre Width</th>
<th>Width</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a/\text{km})</td>
<td>26554.3</td>
<td>37.2321</td>
</tr>
<tr>
<td>(e)</td>
<td>0.72271</td>
<td>0.00045</td>
</tr>
<tr>
<td>(i/\text{deg})</td>
<td>63.4349</td>
<td>0.12941</td>
</tr>
<tr>
<td>(T_3/\text{year})</td>
<td>1.30170</td>
<td>1.25017</td>
</tr>
</tbody>
</table>

** The phase structure of the intermediate period motion is determined by two parameters: \(y_1 = L - H/2\) and \(x_2 = g\). In this example, \(y_1\) is fixed as in Eq.(12) and \(x_2\) is fixed at \(\pm 90^\circ\). The libration width and the centre period are calculated by Eq.(24).

4. Long period motion: qualitative analyses

To show the great difference between the classical critical inclination problem and that combined with mean motion resonance, we first analyse the problem qualitatively.

In the first place, the \(J_2^2\) terms are the dominant perturbations in the classical critical inclination problem and the Hamiltonian (Kozai, 1961):

\[
\mathcal{H} = -\frac{\mu^2}{2L^2} - \frac{\mu^4 R_e^2 J_2}{L^6} \left( \frac{3}{4} \frac{H^2}{G^2} - 1 \right) \left( \frac{L}{G} \right)^3 + \frac{\mu^6 R_e^4 J_2^2}{L^{10}} \frac{3}{16} \left( \frac{L^5}{G^5} - \frac{L^7}{G^7} \right) \left( 1 - 16 \frac{H^2}{G^2} + 15 \frac{H^4}{G^4} \right) \cos 2g
\]  

(25)

determines a one-degree-of-freedom system with the variables \((G, g)\). We reduce this system to a standard pendulum model by expanding \(\mathcal{H}\) with respect to \(G\) at its critical value \(G^*\):

\[
\mathcal{H}^*(G, g) = \frac{1}{2} \mathcal{H}_1''(G^*; J_2) (G - G^*)^2 + \mathcal{H}_2(G^*; J_2^2) \cos 2g
\]  

(26)

with \(\mathcal{H}_1''\) being the second derivative of \(\mathcal{H}_1\) with respect to \(G\), and:

\[
\begin{align*}
\mathcal{H}_1 &= -\frac{\mu^4 R_e^2 J_2}{L^6} \left( \frac{3}{4} \frac{H^2}{G^2} - 1 \right) \left( \frac{L}{G} \right)^3 \\
\mathcal{H}_2 &= \frac{\mu^6 R_e^4 J_2^2}{L^{10}} \frac{3}{16} \left( \frac{L^5}{G^5} - \frac{L^7}{G^7} \right) \left( 1 - 16 \frac{H^2}{G^2} + 15 \frac{H^4}{G^4} \right) \cos 2g
\end{align*}
\]
The libration width is then calculated by

$$\Delta G_I = 2 \sqrt{\frac{\mathcal{H}_2(G^*)}{\mathcal{H}_1''(G^*)}}$$  \hspace{1cm} (27)

In the second place, the resonant tesseral harmonics dominate the problem for a Molniya orbit. By choosing $J_{22}$ as the largest harmonics, the Hamiltonian

$$\mathcal{H} = -\frac{\mu^2}{2L^2} - \omega_e H - \frac{\mu^4}{L^6} \frac{R_e J_2}{4} \left( \frac{3H^2}{G^2} - 1 \right) \left( \frac{L}{G} \right)^3 - B_{2,2,0,-1}(a, e, i) \cos [2(x_3 - \lambda_{22}) + 2g]$$  \hspace{1cm} (28)

determines a double resonance system, where $x_3$ is the resonant angle defined in Eq. (1), $B_{2,2,0,-1}$ is the function of $a, e, i$ defined in the Appendices (see Eq. A.3). The motion of $x_3$ and the argument of perigee $g$ can be separated because the former varies much faster than the latter. Under the assumption that $x_3$ librates around its centre $\lambda_{22}$, we just fix it, i.e. let $x_3 = \lambda_{22}$. This is not a rigorous derivation, but it is acceptable for qualitative analyses. Then a reduced one-degree-of-freedom system, with variables being $(G, g)$ as in the classical case, is obtained. Its reduction to a pendulum model is obtained by applying the previous strategy:

$$\mathcal{H}^*(G, g) = \frac{1}{2} \mathcal{H}''_1(G^*)(G - G^*)^2 - B_{2,2,0,-1}(a^*, e^*, i^*) \cos 2g$$  \hspace{1cm} (29)

with the libration width given by

$$\Delta G_{II} = 2 \sqrt{\frac{B_{2,2,0,-1}(a^*, e^*, i^*)}{\mathcal{H}_1''(G^*)}}$$  \hspace{1cm} (30)

The difference between $\Delta G_I$ and $\Delta G_{II}$ is significant to understand the critical inclination problem of the Molniya orbits. Let the orbital elements be $a = 26,560km$, $e = 0.72$ and $i = 63.4^\circ$, then $|\Delta G_I|/|\Delta G_{II}| \approx 0.1$. It shows that the perigee resonance caused by the critical inclination is enhanced by the 2:1 mean motion resonance with the Earth rotation.

5. Long period motion: the phase structure

When $\mathcal{H}_1$ is taken into consideration, $h_0$ is no longer a constant and $h_0 = h - (\frac{1}{2}w_{22}p_2^2 + w_{23}p_2p_3)$, in which $h$ is the Hamiltonian constant.
5.1. Definition of the section surface

In the analytic solution of the intermediate period motion, there exists an variable $u$, which is linear with the time. And the solution is $4K$-periodic with respect to $u$. Then it is reasonable to define the section surface, a two dimensional subspace in the four dimensional phase space, by

$$\mathcal{H} = h, \quad u = K(\kappa), \quad 0 < \kappa < 1$$

(31)

where $\kappa = \kappa_1$ or $\kappa = \kappa_2$. Specifically, the section surface is defined in two cases:

1. If $0 \leq \kappa_1 \leq 1$ or $|h_0| < \alpha$, the intermediate period motion librates. The equation of the section surface, followed by Eq.(31), takes the form of

$$x_3 = \theta + \arcsin(\kappa_1), \quad y_3 = y_3^c$$

(32)

where $\kappa_1$ is defined in Eq. (17) and $h_0 = h - \frac{1}{2}w_{22}p_2^2$ in this case.

2. If $\kappa_1 > 1$ or $h_0 < -\alpha$, the intermediate period motion circulates. The equation of the section surface takes the following form

$$x_3 = \theta + \frac{\pi}{2}, \quad y_3 = y_3^c + \frac{w_{23}}{-w_{33}}p_2 + \sqrt{\left(\frac{w_{23}}{-w_{33}}\right)^2 + \frac{2(h + \alpha) - w_{22}p_2^2}{w_{33}}}$$

(33)

There is a problem to determine the circulating or librating status of the intermediate period motion. $h_0$ is a constant if no perturbation is introduced, and the previous classification is well-defined. When $\mathcal{H}_1$ is added, the Hamiltonian constant becomes $h$, and $h_0 = h - \frac{1}{2}w_{22}p_2^2 + w_{23}p_2p_3$. The problem is whether or not the type of the intermediate period motion can be classified according to $h_0$, no longer a constant. When the time is intermediate, i.e. several years, the variance of $h_0$ is so small that can be seen as unchanged. Then it is possible to use $h_0$ to classify the intermediate period motion.

Actually, considering that $-2.8 \times 10^{-8} < \frac{1}{2}w_{22}p_2^2 < 0$ and $0 < w_{23}p_2p_3 < 6.2 \times 10^{-10}$ in the area of interest, $h < -\alpha - 2.9 \times 10^{-8}$ implies $h_0 < -\alpha$ (i.e. the intermediate period motion circulates) and $-\alpha + 6.3 \times 10^{-10} < h < \alpha - 2.9 \times 10^{-8}$ implies $|h_0| < \alpha$ (i.e. the intermediate period motion librates). In addition, $\alpha$ is a function of $x_2$, with the maximum and minimum values being:

$$\begin{align*}
\alpha_{\text{min}} &= 6.8354 \times 10^{-8}, \quad \text{when } x_2 = 165.72^\circ \\
\alpha_{\text{max}} &= 9.6926 \times 10^{-8}, \quad \text{when } x_2 = 256.97^\circ
\end{align*}$$

Hence we classify the intermediate period motion, by the constant $h$, in the following three cases:
A: When $-6.7 \times 10^{-8} < h < 3.9 \times 10^{-8}$, the intermediate period motion always librates.

B: When $h < -1.26 \times 10^{-7}$, the intermediate period motion always circulates.

C: When $-1.26 \times 10^{-7} < h < -6.7 \times 10^{-8}$, the intermediate period motion sometimes librates and sometimes circulates depending on the values of $h$ and $\alpha(x_2)$. Also, chaos may occur in this case when $|h_0 + \alpha|$ is sufficiently small.

We would not discuss the case of $h > 3.9 \times 10^{-8}$. In this region, it is possible that there is no solution for the intermediate period motion, because $h_0$ might be larger than $\alpha$. If the solution exists, it is the same as the Case A.

5.2. The first integral $\bar{I}_3$

By applying the Von-Zeipel’s transformation to the perturbed Hamiltonian system determined by $H = H_0 + H_1$ (see Eq. 16), an independent first integral is obtained. This is also available by using the Lie transformation (Henrard, 1990; Delhaise and Henrard, 1993). We introduce the Delhaise’s result directly and the first integral $\bar{I}_3$ takes the following form:

$$\bar{I}_3 = I_3 - \left( \frac{\partial I_3}{\partial h_0} \right) (\bar{H}_1 - H_1) + \cdots$$

(34)

$I_3$ is the first integral of the unperturbed system (determined by $H_0$), defined as

$$I_3(x_2, p_2; h_0) = \frac{1}{2\pi} \int p_3(x_3, h_0; x_2, p_2) \, dx_3$$

(35)

and $\bar{H}_1$ is the averaged perturbed part with the following form:

$$\bar{H}_1(x_2, p_2; h) = \frac{1}{T_3} \int_0^{T_3} H_1(x_2, p_2(t), x_3(t), p_3(t); h) \, dt$$

$$= w_{23} p_2 p_3 + \frac{1}{2} w_{22} p_2^2$$

(36)

In the first equation of Eq. (36), $p_2$, $x_3$ and $p_3$ are functions of the time, and they can be approximated by the intermediate period motion (see Eqs. 19-22), considering that the total integration time is $T_3$. As a result, $\bar{I}_3$ is derived analytically rather than semi-numerically in the Delhaise’s work (Delhaise and Henrard, 1993).
5.3. Trajectory in the section surface

$I_3$ is the equation of the trajectories of the long period motion on the section surface, which defines $y_2$ as a function of $x_2$, with $h$ being the parameter.

**Case A: The intermediate period motion librates**

In this case, the section surface is obviously defined by Eq.(32). Restricted in the section surface, $I_3$ is a function of $x_2$ and $p_2$ with $h$ being the parameter. We expand $I_3(x_2, p_2; h)$ with respect to $p_2$ at $p_2 = 0$ up to the third order, i.e.

$$I_3(x_2, p_2; h) = I_3^{(0)} + I_3^{(1)} p_2 + \frac{1}{2} I_3^{(2)} p_2^2 + \mathcal{O}(p_2^3)$$

where $I_3^{(0)} = I_3(x_2, p_2 = 0; h)$, $I_3^{(1)} = 0$ and $I_3^{(2)}$ are the first and second order partial derivatives of $I_3(x_2, p_2; h)$ with respect to $p_2$ at $p_2 = 0$. Then the trajectories in the section surface are determined by the constants $C_3$ and $h$, through $I_3 = C_3$ and $\mathcal{H} = h$, and the explicit expression is

$$p_2^2 = 2 \frac{C_3 - \left[ I_3(x_2, 0; h) - \left( \frac{\partial I_3}{\partial h_0} \right) \mathcal{H}_1(x_2, 0; h) \right]}{-w_{22} \left[ \left( \frac{\partial I_3}{\partial h_0} \right) - \frac{\partial^2 I_3}{\partial h_0^2} \mathcal{H}_1(x_2, 0; h) \right]} \triangleq f(x_2; h, C_3)$$

The expressions of $I_3$, $\frac{\partial I_3}{\partial h_0}$, $\frac{\partial^2 I_3}{\partial h_0^2}$ and $\mathcal{H}_1$ are listed in the Appendices (see Eqs. B.1–B.3).

Three basic types of the phase structures of the long period motion are discovered in case A (i.e. $-6.7 \times 10^{-8} < h < 3.9 \times 10^{-8}$):

(I) Let $h = 0$, the phase structure of the long period motion in the section surface is obtained and illustrated in Fig. 1.

(II) Let $h = -6.5 \times 10^{-8}$, the phase structure is obtained and illustrated in Fig. 2.

(III) Let $h = -5.0 \times 10^{-8}$, the phase structure, illustrated in Fig. 3, is of the transitional type, between type I and type II. Especially, when $h = -4.5 \times 10^{-8}$ or $h = -5.938 \times 10^{-8}$, the two unstable equilibria are connected. These correspond to the critical situations, illustrated in Fig. 4.

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Figure 1: The phase structure of the long period motion in the section surface (Case A, type I). Let the Hamilton constant $h = 0$, which means that the intermediate period motion always librates.
Figure 2: The phase structure of the long period motion in the section surface (Case A, type II). Let the Hamilton constant $h = -6.5 \times 10^{-8}$, which means that the intermediate period motion always librates.
Figure 3: The phase structure of the long period motion in the section surface (Case A, type III). Let the Hamilton constant $h = -5.0 \times 10^{-8}$, which means that the intermediate period motion sometimes librates and sometimes circulates depending on the evolution of $x_2$. 
Figure 4: The phase structures of the long period motion in the section surface (Case A, type III). These correspond to the critical situations, in which the two unstable equilibria are connected.

(a) \( h = -4.500 \times 10^{-8} \)

(b) \( h = -5.938 \times 10^{-8} \)
For convenience, the Keplerian elements, instead of \((x_2, p_2)\), are used in the figures, similarly hereinafter. In both types I and II (see Figs. 1, 2), three kinds of curves are observed: (1) libration around one of the stable equilibria, (2) double libration around both stable equilibria and the inner unstable equilibrium and (3) circulation. The libration widths for each figure are discussed below for clarity.

The difference between the type I and II is the stabilities of the equilibria. For the type I, the stable equilibria locate at \(x_2 \approx 90^\circ\) and \(x_2 \approx 270^\circ\), the inner unstable equilibrium locates at \(x_2 \approx 0^\circ\) and the outer unstable equilibrium locates at \(x_2 \approx 180^\circ\). For the type II, however, the stable equilibria locate at \(x_2 \approx 0^\circ\) and \(x_2 \approx 180^\circ\) and the inner and outer unstable equilibria locate at \(x_2 \approx 90^\circ\) and \(x_2 \approx 180^\circ\), respectively.

The type III is the type of transition. As \(h\) varies, the phase structures change from the type I to the type II. Take \(h = -5.0 \times 10^{-8}\) for example, Fig. 3 illustrates that there are four stable equilibria, one outer and three inner unstable equilibria for the type III phase structure. Here only the \(\omega - i\) structure is illustrated for simplicity.

Two critical situations of the type III are obtained to show how the transition takes place (see Fig. 4). If \(h = -4.5 \times 10^{-8}\) or \(h = -5.938 \times 10^{-8}\) (see Fig. 4), the two unstable equilibria are connected, i.e. the differentiation of the ‘inner’ and the ‘outer’ equilibria are meaningless. The upper one \((h = -4.5 \times 10^{-8}\), Fig. 4(a)) is similar to the type I (see Fig. 1) while the lower one \((h = -5.938 \times 10^{-8}\), Fig. 4(b)) is similar to the type II (see Fig. 2), except that the double libration does not exist.

The two critical values of \(h\) give the upper and lower bounds of the type III. Specifically, \(h > -4.5 \times 10^{-8}\) implies that the phase structures are of the type I, \(h < -5.938 \times 10^{-8}\) implies that the phase structures are of the type II, and the type III corresponds to \(-5.938 \times 10^{-8} < h < -4.5 \times 10^{-8}\).

**Case B: The intermediate period motion circulates**

In this case, it is possible to expand \(\bar{I}_3\) with respect to \(p_2\) as Eq.(37). However, since \(p_3 \neq 0\) in the section surface, the analytic expansion is too cumbersome. As a result, we apply central algorithm to calculate the numerical derivatives of \(\bar{I}_3\), i.e. \(\bar{I}_3^{(1)}\) and \(\bar{I}_3^{(2)}\). The expressions of \(I_3\), \(\frac{\partial I_3}{\partial h_0}\), and \(\mathcal{H}_1\) are listed in the Appendices (Eqs.B.4–B.6).

Let \(h = -1.5 \times 10^{-7}\), the phase structure of the long period motion in the section surface is obtained and illustrated in Fig. 5. The phase structure
is the same as the type II in case A.

**Case C: The type of the intermediate period motion depends on** $x_2$

In this case, the numerical derivative method fails for $h_0 + \alpha \approx 0$. Instead, $\bar{I}_3(x_2, y_2; h) = C_3$ is applied directly as the implicit equation of the long term trajectories. $y_2$ is calculated for each $x_2$ by the method of bisection.

Let $h = -8.0 \times 10^{-8}$, the phase structure of the long period motion in the section surface is obtained and illustrated in Fig. 6, in which the thicker lines indicate that the intermediate period motion librates and the thinner lines indicate it circulates. The phase structure is similar to that of the type II in case A. Anomalies are detected when the intermediate period motion is near the separatrix, i.e. $h_0 + \alpha \approx 0$.

**The libration widths**

The libration widths (half-width) for each chosen $h$ are obtained from the phase figures.

- $h = 0$, (Case A, type I, see Fig. 1): $\Delta i \approx 0.5^\circ$, $\Delta e \approx 0.02$
- $h = -5.0 \times 10^{-8}$, (Case A, type III, see Fig. 3): $\Delta i \approx 0.19^\circ$
- $h = -6.5 \times 10^{-8}$, (Case A, type II, see Fig. 2): $\Delta i \approx 0.27^\circ$, $\Delta e \approx 0.007$
- $h = -8.0 \times 10^{-8}$, (Case C, type II, see Fig. 6): $\Delta i \approx 0.5^\circ$, $\Delta e \approx 0.012$
- $h = -1.5 \times 10^{-7}$, (Case B, type II, see Fig. 5): $\Delta i \approx 0.3^\circ$, $\Delta e \approx 0.007$

It is seen that the libration widths in the inclination and the eccentricity are about $0.5^\circ$ and $0.02$, respectively, for a typical Molniya orbit (i.e. the resonant angle $x_3$ librates in the vicinity of its centre, and so does the perigee).
Figure 5: The phase structure of the long period motion in the section surface (Case B, type II). Let the Hamilton constant $h = -1.5 \times 10^{-7}$, which means that the intermediate period motion always circulates.
Figure 6: The phase structure of the long period motion in the section surface (Case C, type II). Let the Hamilton constant $h = -8.0 \times 10^{-8}$. Thicker lines indicate that the intermediate period motion librates while the thinner lines indicate it circulates.
5.4. Numerical tests

Numerical evolution of the two-degree-of-freedom system determined by the Hamiltonian $\mathcal{F}$ (Eq. 8) are applied to verify the results of the phase structures of the long period motion.

Figs. 7 – 11 illustrate the three types of the phase structures of case A and the phase structures of case B and C. The Hamiltonian constants $h$ are taken exactly the same values as in Figs. 1, 2, 3 and 5. Only $\omega - i$ curves are illustrated for simplicity. The intermediate period motion is not removed in the numerical integration. The total integration time for each curve is 500 to 2000 years, depending on the period of the long period motion. As showed in the figures, the numerical results verify the three types of phase structures of the long period motion and verify the values of the libration widths for the chosen $h$. In Fig. 10, anomalies occur when the intermediate period motion jumps from libration to circulation or vice versa.

![Figure 7](image_url)

Figure 7: The numerical trajectories of the Hamiltonian system determined by $\mathcal{F}$ (see Eq. 8). Let $h = 0$, to be compared with Fig. 1.
Figure 8: The numerical trajectories of the Hamiltonian system determined by $\mathcal{F}$ (see Eq. 8). Let $h = -5 \times 10^{-8}$, to be compared with Fig. 3.

Figure 9: The numerical trajectories of the Hamiltonian system determined by $\mathcal{F}$ (see Eq. 8). Take $h = -6.5 \times 10^{-8}$, to be compared with Fig. 2.
Figure 10: The numerical trajectories of the Hamiltonian system determined by $\mathcal{F}$ (see Eq. 8). Let $h = -8.0 \times 10^{-8}$, to be compared with Fig. 6.

Figure 11: The numerical trajectories of the Hamiltonian system determined by $\mathcal{F}$ (see Eq. 8). Let $h = -1.5 \times 10^{-7}$, to be compared with Fig. 5.
6. Conclusion and discussion

6.1. Conclusion

Following Henrard and Delhaise’s work (Henrard, 1990; Delhaise and Henrard, 1993), the authors study the long term evolution of the Molniya orbits. The evolution problem is reduced to the dynamics of a near separable two-degree-of-freedom Hamiltonian system. In general, the reduced problem is solved by the semi-numerical method presented by Henrard (Henrard, 1990). In the special case of the Molniya orbits as presented in this paper, however, the analytic method can be applied, since the intermediate period motion can be well approximated by the pendulum model.

With the help of analytic solution of the pendulum model, the expression of the first integral, \( \bar{I}_3 \), is obtained analytically, which gives the implicit equation of the trajectories of the long period motion. A series of phase figures are obtained, with the Hamiltonian constant \( h \) being the parameter, from which three types of phase structures are observed. Among the three types, types I and II have been illustrated in Delhaise’s work (Delhaise and Henrard, 1993), and the transitional type III is firstly presented in our work.

In addition, we specify the effect of the mean motion resonance with the Earth rotation on the critical inclination problem. In the double resonance problem, the resonant tesseral harmonics dominate the critical inclination problem, instead of the \( J^2_2 \) term dominating in the case of classical critical inclination problem. Hence, the libration widths of the inclination and the eccentricity in the long period motion are enlarged by those resonant tesseral terms, compared with the case of non-existence of the mean motion resonance.

6.2. Discussion

The work presented in this paper is based on the assumption that the two-degree-of-freedom Hamiltonian system is near separable, considering that the two frequencies in the system differ greatly. This method is reasonable except for the chaotic motion. In section 3, the formula of the period \( T_3 \) is presented in terms of the Jacobi elliptic integral \( K(\kappa) \) in two cases: libration or circulation. It shows that \( T_3 \) is of the magnitude of several years for most situations, and that \( T_3 \) tends to infinity if \( \kappa \) tends to 1 or \( h_0 \) tends to \(-\alpha\), which means that it is not reasonable to separate the Hamiltonian system according to the frequencies if \( h_0 + \alpha \) is sufficiently small. Specifically, \( T_3 \) is larger than 10 years only if \( 1.0 - 10^{-10} < \kappa < 1.0 + 10^{-10} \). This range of
the parameter $\kappa$ is quite small and thus we have not studied much on the situation of chaos.

Luni-solar perturbations are not introduced in this paper. Studies on the effects of third-body attraction on a Molniya orbit are under progress and will be presented in the future.

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Appendix A. Expressions in force model

The expressions of $A(L, G, H)$, $C(L, G, H)$ and $B_{imp}\eta(a, e, i)$ are given as follows (Delhaise and Henrard, 1993).

\[
A(L, G, H) = -\frac{3}{16} \left( \frac{L^5}{G^5} - \frac{L^7}{G^7} \right) \left( 1 - \frac{16H^2}{G^2} + \frac{15H^4}{G^4} \right) \quad (A.1)
\]

\[
C(L, G, H) = \frac{15}{32} \frac{L^5}{G^5} \left( 1 - \frac{18H^2}{5G^2} + \frac{H^4}{G^4} \right) + \frac{3}{8} \frac{L^6}{G^6} \left( 1 - \frac{3H^2}{G^2} \right)^2 \quad (A.2)
\]

\[
B_{imp}\eta(a, e, i) = \frac{\mu J_{lm}}{a} \left( \frac{R_e}{a} \right)^l F_{imp}(i)G_{lpq}(e) \quad (A.3)
\]

where $L, G, H$ are Delaunay variables, $a, e, i$ are Keplerian elements, $J_{lm}$ is the coefficient of the Earth $(l, m)$ spherical harmonic, $R_e$ is the mean radius of the Earth equator, $\mu$ is the product of Newtonian constant of gravitation by the mass of Earth. Functions $F$ and $G$ are the inclination function and the eccentricity function defined by Kaula (Kaula, 1961).

Appendix B. Expressions in solutions

The expressions of $I_3$, $\frac{\partial I_3}{\partial h_0}$, $\frac{\partial^2 I_3}{\partial h_0^2}$ and $\tilde{H}_1$ are listed in the followings, in two cases:
Case A: The intermediate period motion librates
In this case, \( 0 < \kappa = \kappa_1 < 1 \) and
\[
\begin{align*}
I_3 &= 4 \frac{\alpha}{\pi} \sqrt{\frac{\alpha}{-w_{33}}} \left[ E(\kappa) - (1 - \kappa^2)K(\kappa) \right] \\
\frac{\partial I_3}{\partial h_0} &= - \frac{K(\kappa)}{\pi \sqrt{-w_{33} \alpha}} \\
\frac{\partial^2 I_3}{\partial h_0^2} &= \frac{1}{4\pi \alpha \sqrt{-w_{33} \alpha}} \frac{E - (1 - \kappa_1^2)K}{\kappa_1^2(1 - \kappa_1^2)}
\end{align*}
\]
with
\[
\begin{align*}
\bar{p}_2^2 &= p_2^2 + \frac{p_2}{\sqrt{-w_{33} \alpha}} \left[ -2\kappa S_1(x_2) + (4E - 2K)C_1(x_2) \right] \\
&+ \frac{1}{-w_{33} \alpha} \left\{ \left( \frac{E - (1 - 2\kappa^2)K}{K} \right)^2 S_1^2(x_2) \right. \\
&+ \kappa(2K - 4E)S_1(x_2)C_1(x_2) \\
&\left. + \left[ \frac{(4E - 2K)^2}{3} + \frac{8\pi^2}{K} Q(q) \right] C_1^2(x_2) \right\}
\end{align*}
\]
\[
\bar{p}_2\bar{p}_3 = \frac{2}{-w_{33}} \frac{E - (1 - \kappa^2)K}{K} S_1(x_2)
\]

Case B: The intermediate period motion circulates
In this case, \( 0 < \kappa = \kappa_2 < 1 \) and
\[
\begin{align*}
I_3 &= 4 \frac{\alpha}{\pi \kappa} \sqrt{\frac{\alpha}{-w_{33}}} E(\kappa), \quad \frac{\partial I_3}{\partial h_0} = - \frac{\kappa K(\kappa)}{\pi \sqrt{-w_{33} \alpha}} 
\end{align*}
\]
\[
\bar{H}_1(x_2, p_2; h) = w_{23}\bar{p}_2\bar{p}_3 + \frac{1}{2} w_{22} \bar{p}_2^2
\]
with

\[
\bar{p}_2^2 = p_2^2 + \frac{p_2}{\kappa \sqrt{-w_{33} \alpha}} \left[ \frac{\pi}{K} S_1(x_2) + 2 \left( 2E - (2 - \kappa^2)K \right) C_1(x_2) \right] \\
+ \frac{1}{-w_{33} \alpha \kappa^2} \left\{ \frac{E}{K} S_1^2(x_2) + \pi \left( \frac{2E - (2 - \kappa^2)K}{K} \right) S_1(x_2) C_1(x_2) \right. \\
+ 8 \left[ \left( 2E - (2 - \kappa^2)K \right)^2 + \frac{8\pi^2}{K} Q(q) \right] C_1^2(x_2) \left\} \right.
\]

\[
\frac{p_2 p_3}{\kappa K} \sqrt{\frac{\alpha}{-w_{33}}} \\
+ \frac{1}{-w_{33} \kappa^2} \left[ \frac{2E}{K} S_1(x_2) + \pi \left( \frac{2E - (2 - \kappa^2)K}{K} \right) C_1(x_2) \right]
\]

(B.6)

In the above expressions (Eqs. B.1–B.6), E and K are the complete Jacobi elliptic integrals, \( S_1 \) and \( C_1 \) are defined before (see Eq.23), \( Q(q) \) is a function of \( q \), defined as follows:

\[
Q(q) = \sum_{m=1}^{+\infty} \left( \frac{q^m}{1 - q^{2m}} \right)^2, \quad q(\kappa) = \exp \left( -\frac{\pi K(\sqrt{1 - \kappa^2})}{K(\kappa)} \right) \quad (B.7)
\]

References


**Table and Figure captions**

Table 1 Equilibria and their stabilities.

Table 2 Libration center, width and center period of intermediate period motion.

Fig. 1 The phase structure of the long period motion in the section surface (Case A, type I). Let the Hamilton constant $h = 0$, which means that the intermediate period motion always librates.

Fig. 2 The phase structure of the long period motion in the section surface (Case A, type II). Let the Hamilton constant $h = -6.5 \times 10^{-8}$, which means that the intermediate period motion always librates.

Fig. 3 The phase structure of the long period motion in the section surface (Case A, type III). Let the Hamilton constant $h = -5.0 \times 10^{-8}$, which means that the intermediate period motion sometimes librates and sometimes circulates depending on the evolution of $x_2$.  

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Fig. 4 The phase structures of the long period motion in the section surface (Case A, type III). These are the two critical situations, the two unstable equilibria are connected.

Fig. 5 The phase structure of the long period motion in the section surface (Case B, type II). Let the Hamilton constant \( h = -1.5 \times 10^{-7} \), which means that the intermediate period motion always circulates.

Fig. 6 The phase structure of the long period motion in the section surface (Case C, type II). Let the Hamilton constant \( h = -8.0 \times 10^{-8} \). Thicker lines indicate that the intermediate period motion librates while the thinner lines indicate it circulates.

Fig. 7 The numerical trajectories of the Hamiltonian system determined by \( F \) (see Eq. 8). Let \( h = 0 \), to be compared with Fig. 1.

Fig. 8 The numerical trajectories of the Hamiltonian system determined by \( F \) (see Eq. 8). Let \( h = -5 \times 10^{-8} \), to be compared with Fig. 3.

Fig. 9 The numerical trajectories of the Hamiltonian system determined by \( F \) (see Eq. 8). Let \( h = -6.5 \times 10^{-8} \), to be compared with Fig. 2.

Fig. 10 The numerical trajectories of the Hamiltonian system determined by \( F \) (see Eq. 8). Let \( h = -8.0 \times 10^{-8} \), to be compared with Fig. 6.

Fig. 11 The numerical trajectories of the Hamiltonian system determined by \( F \) (see Eq. 8). Let \( h = -1.5 \times 10^{-7} \), to be compared with Fig. 5.
Highlights

1. The effect of the 2:1 mean motion resonance with the Earth rotation on the critical inclination problem is specified.

2. A first integral is obtained analytically, and its level curves present the phase structure of the long period motion.

3. Three types of phase structures are observed, and one of which is first presented.