Integral points for groups of multiplicative type

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Received 12 May 2011; accepted 24 September 2012
Available online 16 October 2012

Communicated by the Managing Editors of AIM

To Professor Kezheng Li on his 65-th birthday

Abstract

We construct a finite subgroup of Brauer–Manin obstruction for detecting the existence of integral points on integral models of homogeneous spaces of linear algebraic groups of multiplicative type. As an application, the strong approximation theorem for linear algebraic groups of multiplicative type is established. Moreover, the sum of two integral squares over some quadratic fields is discussed.

MSC: 11E72; 11G35; 11R37; 14F22; 14G25; 20G30; 20G35

Keywords: Integral point; Linear algebraic group of multiplicative type; Galois cohomology; Brauer–Manin obstruction; Strong approximation; Sum of two squares

0. Introduction

The integral points on homogeneous spaces of semi-simple and simply connected linear algebraic groups of non-compact type were studied by Borovoi and Rudnick in [1] and homogeneous spaces of connected semi-simple linear algebraic groups of non-compact type by Colliot-Thélène and the second author in [2] using the strong approximation theorem and the Brauer–Manin obstruction. Recently, Harari [5] showed that the Brauer–Manin obstruction...
accounts for the nonexistence of integral points. Colliot-Thélène noticed that a finite subgroup of the Brauer group is enough to account for the nonexistence of integral points by compactness arguments. These results are nonconstructive: they do not say which finite subgroup to use. In our previous paper [16], we construct such finite groups for multi-norm tori explicitly. In this paper, we extend this construction for linear algebraic groups of multiplicative type.

The paper is organized as follows. In Section 1, we give an explicit construction of the so-called admissible groups (see Definition 1.9) and establish a criterion for the existence of the integral points in terms of admissible groups. In Section 2, we translate such admissible groups into the finite Brauer–Manin obstruction. As an application, we establish the strong approximation theorem for a group of multiplicative type in Section 3. In Section 4, we discuss the sum of two integral squares over some quadratic fields.

Notation and terminology are standard if not explained. Let $F$ be a number field, $\mathcal{O}_F$ be the ring of integers of $F$, $\Omega_F$ be the set of all primes in $F$ and $\infty_F$ be the set of all infinite primes in $F$. For simplicity, we write $p < \infty_F$ for $p \in \Omega_F \setminus \infty_F$. For any finite set $S_0 \subseteq \Omega_F \setminus \infty_F$, the ring of $S_0$-integers of $F$ is defined as

$$o_{S_0} = \{ x \in F : |x|_p \leq 1 \text{ for all } p < \infty_F \text{ and } p \not\in S_0 \}.$$

Let $F_p$ be the completion of $F$ at $p$ and $o_{F_p}$ be the local completion of $o_F$ at $p$ for each $p \in \Omega_F$. Write $o_{F_p} = F_p$ for $p \in \infty_F$. We also denote the adele ring (resp. the idele group) of $F$ by $\mathbb{A}_F$ (resp. $\mathbb{I}_F$) and

$$F_\infty = \prod_{p \in \infty_F} F_p.$$

Let $\bar{F}$ be a fixed algebraic closure of $F$. A group $G$ of multiplicative type over $F$ is defined as a closed subgroup of a torus over $F$. Let $\hat{G} = \text{Hom}_F(G, \mathbb{G}_m)$ be the character of $G$. Then the functor $G \mapsto \hat{G}$ is an anti-equivalence between the category of $F$-groups of multiplicative type and the category of finitely generated abelian groups with the continuous action of $\text{Gal}(\bar{F}/F)$ (see [13]). Moreover, $G$ is a torus if and only if $\hat{G}$ is free. For any positive integer $k$, the finite group scheme $G[k]$ over $F$ stands for the kernel of the multiplication by $k$ over $\hat{G}$.

Let $X$ be a separated $o_F$-scheme of finite type whose generic fiber $X_F$ is a trivial torsor of $G$. The obvious necessary condition for $X(o_F) \neq \emptyset$ is

$$\prod_{p \in \Omega_F} X(o_{F_p}) \neq \emptyset \quad (0.1)$$

which is assumed throughout this paper. The Brauer group $Br(X_F)$ of $X_F$ is defined as

$$Br(X_F) = H^2_{\text{et}}(X_F, \mathbb{G}_m) \quad \text{and} \quad Br_1(X_F) = \ker[Br(X_F) \to Br(\bar{X})]$$

where $\bar{X} = X_F \times_F \bar{F}$. Since the image of $Br(F)$ induced by the structure morphism lies in $Br_1(X_F)$, one defines

$$Br_d(X_F) = \text{coker}[Br(F) \to Br_1(X)].$$

For any subset $s$ of $Br_d(X_F)$, one can define the integral Brauer–Manin set with respect to $s$ as (see [2])

$$\left( \prod_{p \in \Omega_F} X(o_{F_p}) \right)^s = \left\{ (x_p) \in \prod_{p \in \Omega_F} X(o_{F_p}) : \sum_{p \in \Omega_F} \text{inv}_p(s(x_p)) = 0, \forall s \in s \right\}.$$
1. Construction of X-admissible groups

In this section, we extend the construction of X-admissible groups in [16] for multi-norm tori to groups of multiplicative type. By using the anti-equivalence between the category of F-tori and the category of free \( \mathbb{Z} \)-modules of finite rank with the continuous action of \( \text{Gal}(\overline{F}/F) \), one has the following lemma which should be standard.

**Lemma 1.1.** Let \( T_1 \) and \( T_2 \) be two tori over \( F \) and \( \phi : T_1 \to T_2 \) be a surjective morphism of tori. Then there is a morphism of tori

\[
\psi : T_2 \to T_1 \quad \text{such that} \quad \phi \circ \psi = [l]
\]

for some positive integer \( l \).

**Proof.** Since the category of \( F \)-tori is anti-equivalent to the category of free \( \mathbb{Z} \)-modules of finite rank with the continuous action of \( \text{Gal}(\overline{F}/F) \), one has the injective \( \text{Gal}(\overline{F}/F) \)-module homomorphism

\[
\hat{\phi}_Q : \hat{T}_2 \otimes_{\mathbb{Z}} \mathbb{Q} \to \hat{T}_1 \otimes_{\mathbb{Z}} \mathbb{Q}.
\]

Since a representation of a compact group over a field of characteristic 0 is semi-simple (Maschke’s Theorem), there is a \( \text{Gal}(\overline{F}/F) \)-module homomorphism

\[
\hat{\psi}'_Q : \hat{T}_1 \otimes_{\mathbb{Z}} \mathbb{Q} \to \hat{T}_2 \otimes_{\mathbb{Z}} \mathbb{Q}
\]

such that \( \hat{\psi}'_Q \circ \hat{\phi}_Q = 1 \). Since \( \hat{T}_1 \) is of finite rank, there is a positive integer \( l \) such that

\[
l \hat{\psi}'_Q(\hat{T}_1) \subseteq \hat{T}_2.
\]

Then the homomorphism from \( T_2 \) to \( T_1 \) associated to \( l \hat{\psi}'_Q \) is as required. \( \square \)

Fix a finite Galois extension \( E/F \) such that \( \text{Gal}(\overline{F}/E) \) acts on \( \hat{G} \) trivially. Let \( M \) be a free \( \mathbb{Z} \)-module of finite rank such that \( f : M \to \hat{G} \) is a surjective homomorphism. One can extend this map to the \( \text{Gal}(\overline{F}/F) \)-module surjective homomorphism

\[
\mathbb{Z}[\text{Gal}(E/F)] \otimes_{\mathbb{Z}} M \to \hat{G}, \quad \sigma \otimes m \mapsto \sigma(f(m))
\]

where \( \text{Gal}(\overline{F}/F) \) acts on \( \mathbb{Z}[\text{Gal}(E/F)] \otimes_{\mathbb{Z}} M \) as follows

\[
\tau \circ (\sigma \otimes m) = \hat{\tau} \sigma \otimes m
\]

for \( \tau \in \text{Gal}(\overline{F}/F) \) and \( \sigma \otimes m \in \mathbb{Z}[\text{Gal}(E/F)] \otimes_{\mathbb{Z}} M \). By the anti-equivalence between the category of multiplicative groups over \( F \) and the category of finitely generated \( \mathbb{Z} \)-modules with the continuous action of \( \text{Gal}(\overline{F}/F) \), there is an \( F \)-torus \( T \) such that

\[
0 \to G \xrightarrow{\lambda} \prod_{1}^{n} R_{E/F}(\mathbb{G}_m) \xrightarrow{\nu} T \to 0
\]

(1.2)

where \( n = \text{rank}_{\mathbb{Z}}(M) \).

Applying Lemma 1.1 to (1.2), one can fix a homomorphism

\[
\mu : T \to \prod_{1}^{n} R_{E/F}(\mathbb{G}_m)
\]

such that \( \nu \circ \mu = [l] \) for some positive integer \( l \).
Since all morphisms are of finite type, we fix a finite subset $S_0 \subset \Omega_F \setminus \infty_F$ such that

1. the groups $G, \prod^n_1 R_{E/F}(\mathbb{G}_m)$ and $T$ over $F$ extend to the group schemes $G, R_{E/F}(\mathbb{G}_m)$ and $T$ over $\mathcal{O}_{S_0}$ such that (1.2) extends to

$$0 \rightarrow \mathcal{G} \rightarrow \prod^n_1 R_{E/F}(\mathbb{G}_m) \rightarrow T \rightarrow 0$$

over $\mathcal{O}_{S_0}$ and

$$R_{E/F}(\mathbb{G}_m)(\mathcal{O}_F) = \mathcal{O}_F^*$$

for $p \not\in S_0$.

2. the trivial torsor $X_F$ of $G$ over $F$ extends to the trivial torsor $X_{S_0} = X \times_{\mathcal{O}_F} \mathcal{O}_{S_0}$ of $G$ over $\mathcal{O}_{S_0}$.

3. the homomorphism $\mu$ also extends to $T \rightarrow \prod^n_1 R_{E/F}(\mathbb{G}_m)$ over $\mathcal{O}_{S_0}$.

4. any $l$-torsion point of $T(\mathcal{O}_F)$ is contained in $T(\mathcal{O}_{F_p})$ for all $p \not\in S_0$.

For the above torus $T$, the class group of $T$ with respect to $S_0$ can be defined as follows

$$\text{cls}(T, S_0) = T(\mathbb{A}_F) \left/ \left[ T(F) + \left( \prod_{p \in S_0} T(F_p) \times \prod_{p \not\in S_0} T(\mathcal{O}_{F_p}) \right) \right] \right.$$ and this group $\text{cls}(T, S_0)$ is finite by Theorem 5.1 in [12]. Denote $h(T, S_0) = \sharp \text{cls}(T, S_0)$.

Let $A$ be a finite group scheme over $F$. One defines

$$\text{III}_F^1(A) = \ker \left[ H^1(F, A) \rightarrow \prod_{p < \infty_F} H^1(F_p, A) \right].$$

**Lemma 1.3.** For any torus $T$ over $F$, there is a positive integer $c = c(F, T)$ such that $\text{III}_F^1(T[k])$ is killed by $c$ for all positive integers $k$.

**Proof.** Let $E$ be a finite Galois extension over $F$ such that

$$T_E := T \times_F E \cong \prod^n_1 \mathbb{G}_m, E$$

for some positive integer $n$. Therefore $T_E[k] = \prod^n_1 \mu_k$. Using the inflation–restriction exact sequence of group cohomology, together with its functoriality properties, one obtains the following commutative diagram

$$\begin{array}{ccc}
H^1(E/F, T(E)[k]) & \rightarrow & H^1(F, T[k]) \\
\downarrow & & \downarrow \\
\prod_{p < \infty_F} H^1(F_p, T[k]) & \rightarrow & \prod_{p < \infty_F} H^1(E_p, T[k]).
\end{array}$$

By (9.1.3) Theorem in [9], one knows that $\text{III}_F^1(T_E[k])$ is killed by 2. If $\mu(E)$ is the order of the set of all roots of unity contained in $E$, then $H^1(E/F, T(E)[k])$ is killed by the greatest common divisor of $\mu(E)$ and $[E : F]$. One can take $c = 2(\mu(E), [E : F])$. \( \square \)

From the above proof, one can further obtain a uniform bound for $\text{III}_F^1(T[k])$ for all positive integers $k$. The above result is good enough for our application.
Lemma 1.4. If $T \cong \prod_{i=1}^{d} \mathbb{G}_m$ for some positive integer $d$, then

$$T(F) \cap \left[ \prod_{p \in F} T(F_p) \times \prod_{p \notin F} [2k]T(F_p) \right] \subseteq [k]T(F)$$

for any positive integer $k$.

Proof. Any element $x \in T(F) \cap \left[ \prod_{p \in F} T(F_p) \times \prod_{p \notin F} [2k]T(F_p) \right]$ can be written as $x = (x_1, \ldots, x_d)$ where $x_i \in F^\times$ and $x_i \in (F^\times)_k$ for all $p < \infty_F$ with $1 \leq i \leq d$. Applying (9.1.3) Theorem in [9] for $\mu_{2k}$, one obtains that $x_i^2 \in (F^\times)^2$ for $1 \leq i \leq d$. Therefore $-1 = (\xi_{2i+1})^k \in (F^\times)^k$ and the proof is complete.

For a general torus $T$, one can fix a splitting field $K/F$ of $T$ such that

$$T_K = T \times_F K \cong \prod_{i=1}^{d} \mathbb{G}_{m,K}$$

for some positive integer $d$. Since $K/F$ is a finite extension, the group of $K$-rational torsion points of $T_K$ is finite. Fix a positive integer $t$ such that $t$ kills all $K$-rational torsion points of $T_K$.

Lemma 1.5. Let $T$ be a torus over $F$ as above and $k$ be a positive integer. Then there is a finite subset $S_1$ of $\Omega_F \setminus \infty_F$ containing $S_0$ such that

$$T(F) \cap \left[ \prod_{p \in S} [c \cdot 2t \cdot h(T, S_0) \cdot k]T(F_p) \times \prod_{p \notin S} T(o_{F_p}) \right] \subseteq [k]T(F)$$

for any finite subset $S \supseteq S_1$, where $c = c(F, T)$ is as in Lemma 1.3 and $t$ is as above.

Proof. Let $U_0$ be a finite subset of $\Omega_K \setminus \infty_K$ such that the ring $o_{U_0}$ of $U_0$-integers of $K$ is the integral closure of $o_{S_0}$ inside $K$. Let $U$ be a finite subset of $\Omega_K \setminus \infty_K$ containing $U_0$ such that

$$T_{U_0}(o_U) = \prod_{i=1}^{n} o_U^x$$

where $T_{U_0} = T \times_{o_{S_0}} o_{U_0}$. By the Dirichlet unit theorem, one obtains that $T_{U_0}(o_U)$ is a finitely generated abelian group. Since $T(o_{U_0}) = T_{U_0}(o_{U_0})$ is a subgroup of $T_{U_0}(o_U)$ and $T(o_{S_0})$ is a subgroup of $T(o_{U_0})$, one concludes that $T(o_{S_0})$ is a finitely generated abelian group. Therefore $T(o_{S_0})/[c \cdot 2t \cdot k]T(o_{S_0})$ is finite.

If the coset $\alpha + [c \cdot 2t \cdot k]T(o_{S_0})$ of $T(o_{S_0})/[c \cdot 2t \cdot k]T(o_{S_0})$ satisfies that

$$\alpha \not\in [c \cdot 2t \cdot k]T(F_p)$$

for some prime $p < \infty_F$, we fix such a prime $p_0$. Let $S$ be the set consisting of all such $p_0$ and $S_1 = S_0 \cup S$. 

For any finite set $S \supseteq S_1$ and
\[
\alpha \in T(F) \cap \left[ \prod_{p \in S} [c \cdot 2t \cdot h(T, S_0) \cdot k] T(F_p) \times \prod_{p \notin S} T(o_{F_p}) \right],
\]
there is $\beta_p \in T(F_p)$ such that $\alpha = [c \cdot 2t \cdot h(T, S_0) \cdot k] \beta_p$ for each $p \in S$. Since $\beta_p$ can be viewed as an element in $T(\mathbb{A}_F)$ whose $p$ component is $\beta_p$ and the others are 1, there is $\omega_p \in T(F)$ such that
\[
\omega_p + [h(T, S_0)] \beta_p \in \left[ \prod_{p \in S_0} T(F_p) \times \prod_{p \notin S_0} T(o_{F_p}) \right]
\]
for each finite $p \in S \setminus S_0$. This implies that
\[
\gamma = \alpha + [c \cdot 2t \cdot k] \left( \sum_{p \in S \setminus (S_0 \cup \infty_F)} \omega_p \right) \in T(o_{S_0}).
\]
Therefore the coset of $\gamma + [c \cdot 2t \cdot k] T(o_{S_0})$ of $T(o_{S_0})/[c \cdot 2t \cdot k] T(o_{S_0})$ is not one of the above mentioned cosets. One concludes that
\[
\alpha \in T(F) \cap \left[ \prod_{p \in \infty_F} T(F_p) \times \prod_{p \notin \infty_F} [c \cdot 2t \cdot k] T(F_p) \right].
\]
Hence
\[
T(F) \cap \left[ \prod_{p \in S} [c \cdot 2t \cdot h(T, S_0) \cdot k] T(F_p) \times \prod_{p \notin S} T(o_{F_p}) \right] \\
\subseteq T(F) \cap \left[ \prod_{p \in \infty_F} T(F_p) \times \prod_{p \notin \infty_F} [c \cdot 2t \cdot k] T(F_p) \right].
\]
For any element
\[
x \in T(F) \cap \left[ \prod_{p \in \infty_F} T(F_p) \times \prod_{p \notin \infty_F} [c \cdot 2t \cdot k] T(F_p) \right],
\]
there is $y \in T(F)$ such that $[c] x = [c \cdot 2t \cdot k] y$ by Lemma 1.3. Let $z = x - [2t \cdot k] y$. Then $z \in T(F)[c]$. By applying Lemma 1.4 to $T_K = T \times_F K$ over $K$, one obtains that there is $w \in T(K)$ such that $z = [t \cdot k] w$. Since $z$ is a torsion point of $T$, one has that $w$ is also a torsion point of $T_K$. Since $t$ kills all $K$-torsion points of $T_K$, one concludes that $z = 0$. The proof is complete. \qed

**Lemma 1.6.** If $C_p$ is an open subgroup of $G(F_p)$ and $k$ is a positive integer, then
\[
\lambda(C_p) \cdot \mu([k]T(F_p))
\]
is an open subgroup of
\[
\prod_{l=1}^{n} R_{E/F}(\mathbb{G}_m)(F_p) = \prod_{l=1}^{n} E_p^\times
\]
for the $p$-adic topology with $p < \infty_F$. 
Proof. Since $C_p$ is an open subgroup of $G(F_p)$ and $\lambda[G(F_p)]$ is a closed subgroup of
\[
\prod_{l=1}^{n} R_{E/F}^{\times} = \prod_{l=1}^{n} E_p^\times,
\]
there is a positive integer $a$ such that
\[
\left[ \prod_{l=1}^{n} (1 + p^a o_{E_p}) \right] \cap \lambda[G(F_p)] \subseteq \lambda(C_p).
\]
Since $\mu \circ \nu$ is a continuous endomorphism of
\[
\prod_{l=1}^{n} R_{E/F}^{\times} = \prod_{l=1}^{n} E_p^\times,
\]
there is a positive integer $a_1 > a$ such that
\[
\mu \circ \nu \left[ \prod_{l=1}^{n} (1 + p^{a_1} o_{E_p}) \right] \subseteq \prod_{l=1}^{n} (1 + p^a o_{E_p}).
\]
By Hensel’s lemma, there is a positive integer $a_2 > a_1$ such that
\[
(1 + p^{a_2} o_{E_p}) \subseteq (1 + p^{a_1} o_{E_p})^{l_k}.
\]
For any $x \in \prod_{l=1}^{n} (1 + p^{a_2} o_{E_p})$, there is
\[
y \in \prod_{l=1}^{n} (1 + p^{a_1} o_{E_p}) \text{ such that } x = y^{l_k}.
\]
Then
\[
x = y^{l_k} = [y^{l_k}(\mu \circ \nu(y^{k}))^{-1}] (\mu \circ \nu(y^{k})).
\]
Since
\[
\nu[y^{l_k}(\mu \circ \nu(y^{k}))^{-1}] = [l] \nu(y^{k}) - \nu \circ \mu \circ \nu(y^{k}) = 0,
\]
one concludes
\[
y^{l_k}(\mu \circ \nu(y^{k}))^{-1} \in \lambda(C_p).
\]
Therefore
\[
\prod_{l=1}^{n} (1 + p^{a_2} o_{E_p}) \subseteq \lambda(C_p) \cdot \mu([k]T(F_p))
\]
and the proof is complete. □

Since $X$ is separated over $o_F$, one can view $X(o_{F_p})$ as an open subset of $X_F(F_p)$ by the natural map for any $p \in \Omega_F$.

Definition 1.7. Define
\[
\text{Stab}(X(o_{F_p})) = \{ g \in G(F_p) : gX(o_{F_p}) = X(o_{F_p}) \}
\]
for $p \in \Omega_F$. 
It is clear that \( \text{Stab}(X(o_{F_p})) \) is an open subgroup of \( G(F_p) \). Since \( X_F \) is a trivial torsor of \( G \) over \( F \), one has that

\[
\text{Stab}(X(o_{F_p})) = G(o_{F_p})
\]

for almost all \( p \) in \( \Omega_F \).

**Definition 1.8.** Define

\[
\text{Stab}_A(X) = \prod_{p \in \Omega_F} \text{Stab}(X(o_{F_p})).
\]

It is clear that \( \text{Stab}_A(X) \) is an open subgroup of \( G(A_F) \). The map given by (1.2)

\[
\lambda : G(A_F) \longrightarrow \prod_{i=1}^n R_{E/F}(\mathbb{G}_m)
\]

induces the homomorphism

\[
\lambda : G(A_F) \longrightarrow \prod_{i=1}^n \mathbb{I}_E.
\]

**Definition 1.9.** An open subgroup \( \Xi \) of \( \prod_{i=1}^n \mathbb{I}_E \) is called \( X \)-admissible if

\[
\lambda[\text{Stab}_A(X)] \subseteq \Xi
\]

and the induced map

\[
\lambda : G(A_F)/G(F)\text{Stab}_A(X) \longrightarrow \prod_{i=1}^n \mathbb{I}_E \bigg/ \left( \prod_{i=1}^n E_i^\times \right) \cdot \Xi
\]

is injective.

The main result of this section is to show the existence of admissible subgroups.

**Theorem 1.10.** If \( X \) is a separated scheme over \( o_F \) of finite type such that the generic fiber \( X_F \) is a trivial torsor of a multiplicative group \( G \) satisfying (1.2), then \( X \)-admissible subgroups of \( \prod_{i=1}^n \mathbb{I}_E \) always exist.

**Proof.** By the conditions (1) and (2) for the choice of \( S_0 \), one has

\[
\lambda[\text{Stab}(X(o_{F_p}))] = \lambda[G(o_{F_p})] = \ker \left[ v : \prod_{i=1}^n o_{E_p}^\times \longrightarrow T(o_{F_p}) \right]
\]

for all \( p \notin S_0 \). For each \( p \in S_0 \), one can fix a positive integer \( r_p \) such that \( r_p \) kills all torsion points of \( T(F_p) \). Let

\[
r = \prod_{p \in S_0} r_p.
\]

Let \( S_1 \) be a finite subset of \( \Omega_F \) outside \( \infty_F \) such that Lemma 1.5 holds for \( k = l \cdot r \). Define

\[
\Xi = \left[ \prod_{p \in S_1} \lambda(\text{Stab}(X(o_{F_p}))) \cdot \mu(\chi \cdot c \cdot 2t \cdot h(T(S_0) \cdot r)[T(F_p)]) \right] \times \left( \prod_{p \notin S_1} \prod_{i=1}^n o_{E_p}^\times \right).
\]
By Lemma 1.6, one has that $\Xi$ is an open subgroup of $\prod I^\Xi$ and
$$\lambda[Stab_\xi(X)] \subseteq \Xi.$$ 

Consider $\sigma \in G(\mathbb{A}_F)$ such that $\lambda(\sigma) = a \cdot i$ with
$$a \in \prod_{1}^{n} E^\times \quad \text{and} \quad i \in \Xi.$$ 

Then
$$v(a) \in \left[ \prod_{p \in S_1} [c \cdot 2t \cdot h(T, S_0) \cdot l \cdot r]T(F_p) \times \prod_{p \not\in S_1} T(o_{F_p}) \right].$$

By Lemma 1.5, there is $u \in T(F)$ such that $v(a) = [l \cdot r](u)$. Since
$$v(a \cdot (\mu([r](u)))^{-1}) = v(a) - [l \cdot r](u) = 0,$$
one obtains that
$$a \cdot (\mu([r](u)))^{-1} \in \lambda(G(F)).$$
Let $i = (i_p)_{p \in \Omega_F}$. One can write
$$i_p = s_p \cdot \mu([c \cdot 2t \cdot h(T, S_0) \cdot r](n_p))$$
with
$$s_p \in \lambda(Stab(X(o_{F_p}))) \quad \text{and} \quad n_p \in T(F_p)$$
for $p \in S_1$. This implies that
$$[l \cdot r](u + [c \cdot 2t \cdot h(T, S_0)](n_p)) = 0$$
for $p \in S_1$. Therefore $u + [c \cdot 2t \cdot h(T, S_0)](n_p)$ is a torsion point of $T(F_p)$ for each $p \in S_1$.

If $p \in S_0$, then $r$ kills $u + [c \cdot 2t \cdot h(T, S_0)](n_p)$. One concludes that
$$\mu([r](u)) \cdot \mu([c \cdot 2t \cdot h(T, S_0) \cdot r](n_p)) = 1.$$ 

If $p \in S_1 \setminus S_0$, one has
$$[r]u + [c \cdot 2t \cdot h(T, S_0) \cdot r](n_p) \in T(o_{F_p})$$
by the condition (4) for the choice of $S_0$. By the conditions (1) and (3) for the choice of $S_0$, one obtains that
$$\mu([r]u) \cdot \mu([c \cdot 2t \cdot h(T, S_0) \cdot r](n_p)) \in \prod_{1}^{n} o_{E_p}^\times.$$ 

Therefore one concludes that
$$\mu([r]u) \cdot i \in \lambda[Stab_\xi(X)] \quad \text{and} \quad \sigma \in G(F)Stab_\xi(X).$$
The proof is complete. \( \square \)

Since $X_F$ is a trivial $G$-torsor over $F$, one can fix a rational point $Q \in X_F(F)$ which induces an isomorphism $X_F \cong G$ as $F$-varieties. Since $X$ is separated over $o_F$, the natural morphism
$$\prod_{p \in \Omega_F} X(o_{F_p}) \rightarrow X_F(\mathbb{A}_F)$$

is injective. Define
\[ f : \prod_{p \in \Omega_F} X(o_{F_p}) \to X_F(\mathbb{A}_F) \cong G(\mathbb{A}_F) \overset{\lambda}{\to} \prod_1^n \mathbb{I}_E. \]

**Corollary 1.11.** Let \( \Xi \) be an \( X \)-admissible subgroup of \( \prod_1^n \mathbb{I}_E \). Then
\[ X(o_F) \neq \emptyset \quad \text{if and only if} \quad f \left[ \prod_{p \in \Omega_F} X(o_{F_p}) \right] \cap \left[ \left( \prod_1^n \mathbb{E}^\times \right) \cdot \Xi \right] \neq \emptyset. \]

**Proof.** Since \( X \) is separated over \( o_F \), one has \( X(o_F) \subseteq X(o_{F_p}) \) for all \( p \in \Omega_F \) and
\[ X(o_F) \subseteq \prod_{p \in \Omega_F} X(o_{F_p}) \]
by the diagonal map. If \( X(o_F) \neq \emptyset \), then
\[ f \left[ \prod_{p \in \Omega_F} X(o_{F_p}) \right] \cap \left[ \left( \prod_1^n \mathbb{E}^\times \right) \cdot \Xi \right] \supseteq f_E[X(o_F)] \cap \left( \prod_1^n \mathbb{E}^\times \right) \neq \emptyset \]
and the necessity follows.

Conversely, there is \( y_A \in \prod_{p \in \Omega_F} X(o_{F_p}) \) such that \( f(y_A) \in \left( \prod_1^n \mathbb{E}^\times \right) \cdot \Xi \).

By **Definition 1.9**, there are \( \varrho \in G(F) \) and \( \sigma_A \in Stab_A(X) \) such that \( y_A = \varrho \sigma_A(Q) \). This implies that
\[ \varrho(Q) = \sigma_A^{-1}(y_A) \in \prod_{p \in \Omega_F} X(o_{F_p}). \]

Therefore \( \varrho(Q) \in X(o_F) \neq \emptyset \) and the proof is complete. \( \square \)

If \( \Xi \) is an \( X \)-admissible subgroup of \( \prod_1^n \mathbb{I}_E \), there is an open subgroup \( \Xi_i \) of \( \mathbb{I}_E \) for each \( 1 \leq i \leq n \) such that
\[ \prod_{i=1}^n \Xi_i \subseteq \Xi. \]

By class field theory, there is a finite abelian extension \( K_{\Xi_i}/E \) such that the Artin map
\[ \psi_{K_{\Xi_i}/E} : \mathbb{I}_E/E^\times \Xi_i \cong Gal(K_{\Xi_i}/E) \quad (1.12) \]
gives the isomorphism for \( 1 \leq i \leq n \). Projecting the image of \( f \) to the \( i \)-th component, one can define
\[ f_i : \prod_{p \in \Omega_F} X(o_{F_p}) \longrightarrow \prod_1^n \mathbb{I}_E \longrightarrow \mathbb{I}_E \]
for \( 1 \leq i \leq n \).
Corollary 1.13. With the notation as above, \( X(\alpha_F) \neq \emptyset \) if and only if there is \( x_A \in \prod_{p \in \Omega_F} X(\alpha_{F_p}) \) such that \( \psi_{K_{\Xi_i}/E}(f_i(x_A)) = 1 \) in \( \text{Gal}(K_{\Xi_i}/E) \) for all \( 1 \leq i \leq n \).

Proof. Since \( \psi_{K_{\Xi_i}/E}(f_i(x_A)) = 1 \) for \( 1 \leq i \leq n \), one has

\[
 f(x_A) \in \left[ \prod_{i=1}^n (E^\times : \Xi_i) \right] \subseteq \left[ \prod_{i=1}^n E^\times \right] \cdot \Xi.
\]

The result then follows by arguing as in the proof of Corollary 1.11. \( \square \)

2. Brauer–Manin obstruction

In this section, we interpret \( X \)-admissible subgroups in terms of Brauer–Manin obstruction. This translation has been done for one dimensional tori in [16]. For the general case, the argument is similar. We keep the same notation as the previous section.

Since \( X_F \) is a trivial torsor of \( G \) over \( F \), the fixed rational \( F \)-point \( Q \) gives a \( \text{Gal}(\bar{F}/F) \)-module homomorphism \( \hat{G} \rightarrow \bar{F}[X]^\times \), where \( \bar{F}[X]^\times \) are the global sections of \( \bar{X} \). Then one obtains a homomorphism

\[
 H^2(E, M) \cong H^2(F, \mathbb{Z}[\text{Gal}(E/F)] \otimes \mathbb{Z} M) \rightarrow H^2(F, \hat{G}) \rightarrow H^2(F, \bar{F}[X]^\times) \tag{2.1}
\]

by Shapiro’s lemma. Applying the Hochschild–Serre spectral sequence (see Theorem 2.20 of Chapter III in [6]) in étale cohomology

\[
 H^p(F, H^q(\bar{X}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X_F, \mathbb{G}_m),
\]

one has

\[
 \phi_Q : H^2(E, M) \rightarrow H^2(F, \bar{F}[X]^\times) \rightarrow Br_1(X_F).
\]

Fix a basis \( \{e_1, \ldots, e_n\} \) of \( M \) such that the projection

\[
 p_i : \prod_{1}^{n} R_{E/F}(\mathbb{G}_m) \rightarrow R_{E/F}(\mathbb{G}_m)
\]

to the \( i \)-th component is given by \( \mathbb{Z}e_i \subseteq M \) for each \( 1 \leq i \leq n \). Then the evaluation of the following morphism by using the fixed rational point \( Q \)

\[
 X_F \xrightarrow{Q} G \xrightarrow{\lambda} \prod_{1}^{n} R_{E/F}(\mathbb{G}_m) \xrightarrow{p_i} R_{E/F}(\mathbb{G}_m)
\]
\( \tag{2.2} \)

at \( \prod_{p \in \Omega_F} X(\alpha_{F_p}) \) is \( f_i \) in Section 1.

Let \( \Xi \) be an \( X \)-admissible subgroup of \( \prod_{1}^{n} \mathbb{I}_E \) and \( \Xi_i \) be an open subgroup of \( \mathbb{I}_E \) such that

\[
 \prod_{i=1}^{n} \Xi_i \subseteq \Xi
\]
for each $1 \leq i \leq n$. Let $K_{\Xi_i}/E$ be a finite abelian extension satisfying (1.12) for $1 \leq i \leq n$. Since

$$H^2(\text{Gal}(K_{\Xi_i}/E), \mathbb{Z}e_i) \cong H^1(\text{Gal}(K_{\Xi_i}/E), \mathbb{Q}/\mathbb{Z}) = \text{Hom}(\text{Gal}(K_{\Xi_i}/E), \mathbb{Q}/\mathbb{Z})$$

is finite, one has that the image of the inflation map composed with $\phi_Q$

$$H^2(\text{Gal}(K_{\Xi_i}/E), \mathbb{Z}e_i) \to H^2(E, \mathbb{Z}e_i) \subseteq H^2(E, M) \xrightarrow{\phi_Q} Br_1(X_F)$$

is finite and denoted by $b(\Xi_i)$.

**Definition 2.3.** For any $X$-admissible subgroup $\Xi$ of $\prod_{1}^{n} \mathbb{I}_E$ and an open subgroup $\Xi_i$ of $\mathbb{I}_E$ with

$$\prod_{i=1}^{n} \Xi_i \subseteq \Xi$$

for each $1 \leq i \leq n$, one defines the finite group $b(\Xi)$ of $Br_1(X_F)$ to be generated by $b(\Xi_i)$ for $1 \leq i \leq n$.

One can reformulate Corollary 1.11 in terms of Brauer–Manin obstruction.

**Theorem 2.4.** Let $\Xi$ be an $X$-admissible subgroup of $\prod_{1}^{n} \mathbb{I}_E$. Then

$$X(o_F) \neq \emptyset \text{ if and only if } \left[ \prod_{p \in \Omega_F} X(o_{F_p}) \right]^{b(\Xi)} \neq \emptyset.$$  

**Proof.** The necessity follows from class field theory and one only needs to show the sufficiency. Applying Galois cohomology to the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

with the trivial action, one obtains

$$\delta_i \text{ Hom}(\text{Gal}(K_{\Xi_i}/E), \mathbb{Q}/\mathbb{Z}) \cong H^2(\text{Gal}(K_{\Xi_i}/E), \mathbb{Z})$$

with $1 \leq i \leq n$. For any $\chi \in \text{Hom}(\text{Gal}(K_{\Xi_i}/E), \mathbb{Q}/\mathbb{Z})$, the cup product gives

$$\xi_i = e_i \cup \delta_i(\chi) \in H^2(\text{Gal}(K_{\Xi_i}/E), \mathbb{Z}e_i) \quad \text{and} \quad \beta_i = \phi_Q(\xi_i) \in b(\Xi_i)$$

with $1 \leq i \leq n$.

Let

$$(x_p)_{p \in \Omega_F} \in \left[ \prod_{p \in \Omega_F} X(o_{F_p}) \right]^{b(\Xi)}$$

and evaluate $\beta_i$ at $(x_p)_{p \in \Omega_F}$. Then

$$\text{inv}_p(\beta_i(x_p)) = \sum_{P \mid p} \text{inv}_p(f_i(x_p) \cup \delta_i(\chi))$$

by (2.1), (2.2), (8.1.4) Proposition and (7.1.4) Corollary in [9], where $P$’s are all primes in $E$ above $p$. One has

$$\sum_p \text{inv}_p(f_i(x_p) \cup \delta_i(\chi)) = 0$$
where $P$ runs over all primes of $E$. This implies
\[
\chi(\psi_{K_{\Xi}/E}(f_i([x_p]_{p\in \Omega_F}))) = 0
\]
for all $\chi \in \text{Hom}(\text{Gal}(K_{\Xi}/E), \mathbb{Q}/\mathbb{Z})$ by (8.1.11) Proposition in [9] and (1.12). Therefore
\[
\psi_{K_{\Xi}/E}(f_i([x_p]_{p\in \Omega_F})) = 0
\]
for $1 \leq i \leq n$. The result follows from Corollary 1.13.

3. Strong approximation

As an application, we prove the strong approximation theorem for groups of multiplicative type. Such strong approximation theorem has been established for algebraic tori by Harari in [5] and generalized to connected reductive groups by Demarche in [4]. We give a different proof for groups of multiplicative type where $G$ is not assumed to be connected. We keep the same notation as that in the previous sections.

By (8.1.16) Corollary in [9], the evaluation gives the pair
\[
G(\mathbb{A}_F) \times H^2(F, \hat{G}) \xrightarrow{\text{inv}} \mathbb{Q}/\mathbb{Z}
\]
(3.1)
such that $G(F)$ is lying in the left kernel of the above pair. Let $G(F_{\infty})^0$ be the set of elements in $G(F_{\infty})$ which are lying in the left kernel of (3.1).

**Theorem 3.2.** $G(F) \cdot G(F_{\infty})^0$ is dense in the left kernel of (3.1).

**Proof.** By the definition of $G(F_{\infty})^0$, one only needs to show that $G(F) \cdot U$ contains the left kernel of (3.1) for any open subgroup $U$ of $G(\mathbb{A}_F)$ which contains $G(F_{\infty})$. Without loss of generality, one can assume that
\[
U = G(F_{\infty}) \times \prod_{p < \infty_F} U_p
\]
where $U_p$ is an open subgroup of $G(F_p)$ for all $p < \infty_F$ such that there is a finite subset $S_0$ of $\Omega_F$ outside $\infty_F$ satisfying conditions (1)–(4) in Section 1 and
\[
U_p = G(o_{F_p})
\]
for $p \notin S_0$. For each $p \in S_0$, one can fix a positive integer $r_p$ such that $r_p$ kills all torsion points of $T(F_p)$. Let $r = \prod_{p \in S_0} r_p$ and $S_1$ be a finite subset of $\Omega_F$ outside $\infty_F$ such that Lemma 1.5 holds for $k = l \cdot r$. Define
\[
\Xi = \left[\prod_{p \in S_1} \lambda(U_p) \cdot \mu([c \cdot 2t \cdot h(T, S_0) \cdot r]T(F_p))\right] \times \left(\prod_{p \notin S_1} \prod_{1}^{n} o_{E_p}^\times\right).
\]
By Lemma 1.6, one has that $\Xi$ is an open subgroup of $\prod_{1}^{n} \mathbb{I}_E$ and $\lambda(U) \subseteq \Xi$. Moreover the induced map
\[
\lambda : G(\mathbb{A}_F)/G(F) \cdot U \longrightarrow \prod_{1}^{n} \mathbb{I}_E \left/ \left(\prod_{1}^{n} E_1^\times\right) \right. \cdot \Xi
\]
(3.3)
is injective by the exact same arguments in Theorem 1.10.
Let $\Xi_i$ be an open subgroup of $\mathbb{I}_E$ for each $1 \leq i \leq n$ such that

$$\bigcap_{i=1}^{n} \Xi_i \subseteq \Xi$$

and $K_{\Xi_i}/E$ be a finite abelian extension such that (1.12) holds for $1 \leq i \leq n$. Then

$$0 \rightarrow \prod_{i=1}^{n} \text{Hom}(\text{Gal}(K_{\Xi_i}/E), \mathbb{Q}/\mathbb{Z}) \rightarrow \prod_{i=1}^{n} \text{Hom}(\text{Gal}(\bar{F}/E), \mathbb{Q}/\mathbb{Z}) \cong H^2(E, M).$$

Let $g = (g_p) \in G(\mathbb{A}_F)$ and $\lambda(g) = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i \in \mathbb{I}_E$ for $1 \leq i \leq n$. By the functoriality of evaluation and (8.1.11) Proposition in [9], one obtains

$$\sum_{i=1}^{n} \chi_i(\psi_{K_{\Xi_i}/E}(\sigma_i)) = \text{inv}(g, \lambda^*(\chi_1, \ldots, \chi_n))$$

by (3.1) for all

$$(\chi_1, \ldots, \chi_n) \in \prod_{i=1}^{n} \text{Hom}(\text{Gal}(K_{\Xi_i}/E), \mathbb{Q}/\mathbb{Z}).$$

If $g$ is in the left kernel of (3.1), then $\sigma_i \in E^\times \Xi_i$ for $1 \leq i \leq n$. Therefore

$$\lambda(g) \in \left( \prod_{i=1}^{n} E^\times \right) \cdot \Xi.$$

The injectivity of (3.3) implies that $g \in G(F) \cdot U$. The proof is complete. \qed

When $G$ is a finite commutative group scheme over $F$, the above result follows from Poitou–Tate (see (8.6.13) in [9]).

**Remark 3.4.** When $F_p = \mathbb{R}$, one defines

$$N_G(F_p) = \{ x + \bar{x} : x \in G(\mathbb{C}) \text{ and } \bar{x} \text{ is the conjugate point of } x \}.$$

When $F_p = \mathbb{C}$, one defines

$$N_G(F_p) = G(F_p).$$

Let

$$N_G(F_\infty) = \prod_{p \in \infty_F} N_G(F_p).$$

By the local duality over $\mathbb{R}$ by (7.2.17) Theorem in [9], one has that

$$N_G(F_\infty) \subseteq G(F_\infty)^0$$

with the finite index. One can expect that $G(F) \cdot N_G(F_\infty)$ is dense in the left kernel of (3.1) by more careful study on archimedean places.

**4. Sum of two squares**

The natural extension of Fermat–Gauss’ theorem about the sum of two squares over $\mathbb{Z}$ to the ring of integers of quadratic field $F$ was already studied by Niven for $F = \mathbb{Q}(\sqrt{-1})$ in [10] and
by Nagell for $F = \mathbb{Q}\left(\sqrt{d}\right)$ where
\[
d = \pm 2, \pm 3, \pm 5, \pm 7, \pm 11, \pm 13, \pm 19, \pm 43, \pm 67, \pm 163
\]
in [7,8]. Both followed Gauss’ original idea. Therefore the class number one is assumed and the results obtained there always satisfy the local–global principle. In this section, we will apply the method developed in the previous sections to study this question and give an example for which the local–global principle is no longer true.

It should be pointed out that our method only produces the idelic class groups of $\mathbb{Q}\left(\sqrt{d}, \sqrt{-1}\right)$ for solving the problem of sum of two squares. In order to get the explicit conditions for the sum of two squares, one needs further to construct the explicit abelian extensions of $\mathbb{Q}\left(\sqrt{d}, \sqrt{-1}\right)$ corresponding to the idelic class groups by class field theory. Such explicit construction is a wide open problem (Hilbert’s 12-th problem) in general. Some special cases can be constructed by the ad hoc method. In his series papers [14,15], the first author gives explicit construction for infinitely many $d$ and solves the sum of two squares over infinitely many quadratic fields.

Let $l$ be a prime with $l \equiv -1 \ mod \ 8$ and $F = \mathbb{Q}\left(\sqrt{-2l}\right)$. We will study the sum of two squares over $\mathcal{O}_F$. Let
\[
E = F\left(\sqrt{-1}\right) \quad \text{and} \quad \Theta = E\left(\sqrt[4]{l}\right).
\]

**Lemma 4.1.** The field $\Theta/E$ is unramified over all primes except the prime above $l$.

**Proof.** Since $2$ is totally ramified in $E/\mathbb{Q}$, there is a unique prime $v$ of $E$ over 2. One only needs to show that $v$ is unramified in $\Theta/E$.

Since $\sqrt{-1} \notin E$, one obtains that $x^4 - l$ is irreducible over $E$ by the Kummer theory. By Hensel’s Lemma, there is $s \in (1 + 4\mathbb{Z}_2)$ such that $s^2 = -l$ over $\mathbb{Z}_2$. Since
\[
E_v = \mathbb{Q}_2\left(\sqrt{-1}, \sqrt{2}\right) = \mathbb{Q}_2(\zeta_8)
\]
where $\zeta_8$ is the primitive 8-th root of unity, one has
\[
x^4 - l = x^4 + s^2 = x^4 - s^2\left(\sqrt{-1}\right)^2 = \left(x^2 - s\sqrt{-1}\right)\left(x^2 + s\sqrt{-1}\right)
\]
over $E_v$. By $\zeta_8^2 = \sqrt{-1}$ and $-1 = (\sqrt{-1})^2$, one concludes that $\Theta/E$ is unramified over $v$. □

It is clear that $l$ is ramified in $F/\mathbb{Q}$ and the unique prime of $F$ over $l$ is also denoted by $l$. Since $l \equiv 3 \ mod \ 4$, one has that $l$ is inert in $E/F$. The unique prime of $E$ above $l$ is denoted by $l$ as well and
\[
E_l = \mathbb{Q}_l\left(\sqrt{-1}, \sqrt{l}\right).
\]

**Lemma 4.2.** If $N_{E_l/F_l}(\xi) = 1$, then the Hilbert symbols over $E_l$
\[
\left(\xi, \sqrt{-1}\right)_l = \left(\xi, -\sqrt{-1}\right)_l = 1.
\]
If $N_{E_l/F_l}(\xi) = -1$, then the Hilbert symbols over $E_l$
\[
\left(\xi, \sqrt{l}\right)_l = \left(\xi, -\sqrt{l}\right)_l = -1.
\]
Proof. If \( N_{E_l/F_l}(\xi) = 1 \), then \( \xi \) is a unit of \( E_l \). Therefore
\[
(\xi, \pm \sqrt{I})_{l} = (\xi, \sqrt{-2I})_{l} \cdot (\xi, \pm \sqrt{-2})_{l} = (N_{E_l/F_l}(\xi), \sqrt{-2I})_{l} = 1
\]
by 63:11a in [11] and (1.5.3) Proposition and (7.1.4) Corollary in [9].

If \( N_{E_l/F_l}(\xi) = -1 \), then
\[
(\xi, \pm \sqrt{I})_{l} = (\xi, \sqrt{-2I})_{l} \cdot (\xi, \pm \sqrt{-2})_{l} = (N_{E_l/F_l}(\xi), \sqrt{-2I})_{l} = (-1, 2I)_{l} = -1
\]
by 63:11a in [11] and (1.5.3) Proposition and (7.1.4) Corollary in [9]. □

Let \( B = o_F + o_F \sqrt{-1} \) be the order of \( E \) and \( K_B \) be the ring class field of \( E \) defined by \( B \).

**Proposition 4.3.** Let \( X \) be the scheme defined by \( x^2 + y^2 = n \) for some non-zero integer \( n \) over \( o_F \). Then \( X(o_F) \neq \emptyset \) if and only if there is \( \omega \in \Omega_F(x_w, y_w) \) such that
\[
\psi_{K_B/E} \left( \tilde{f}_E \left( \prod_{w \in \Omega_F} (x_w, y_w) \right) \right) = 1 \quad \text{and} \quad \psi_{\Theta/E} \left( \tilde{f}_E \left( \prod_{w \in \Omega_F} (x_w, y_w) \right) \right) = 1
\]
where \( K_B \) and \( \Theta \) are defined as above, both \( \psi_{K_B/E} \) and \( \psi_{\Theta/E} \) are the Artin maps and
\[
\tilde{f}_E((x_w, y_w)) = \begin{cases} 
(x_w + y_w \sqrt{-1}, x_w - y_w \sqrt{-1}) & \text{if } w \text{ splits in } E/F \\
(x_w + y_w \sqrt{-1}) & \text{otherwise.}
\end{cases}
\]

Proof. Define \( E_w = E \otimes_F F_w \) for any \( w \in \Omega_F \). Let \( B_w \) be the completion of \( B \) inside \( E_w \) for \( w < \infty_F \) and \( B_w = E_w \) for \( w \in \infty_F \). Let
\[
SO_{\mathbb{A}}(B) = \{ \sigma \in R^1_{E/F}(\mathbb{G}_m)(\mathbb{A}_F) : \sigma B = B \}.
\]

By Lemmas 4.1 and 4.2, the natural group homomorphism
\[
\lambda_E : R^1_{E/F}(\mathbb{G}_m)(\mathbb{A}_F)/R^1_{E/F}(\mathbb{G}_m)(F)SO_{\mathbb{A}}(B)
\]
\[
\rightarrow [\mathbb{I}_E/(E^x N_{\Theta/E}([\mathbb{I}_E]))] \times \left[ \mathbb{I}_E / \left( E^x \prod_{w \in \Omega_F} B_w^x \right) \right]
\]
is well-defined.

If \( u \in ker \lambda_E \), there are
\[
\alpha \in E^x \quad \text{and} \quad i \in \prod_{w \in \Omega_F} B_w^x
\]
with \( \lambda_E(u) = \alpha i \). Therefore
\[
N_{E/F}(\alpha) = N_{E/F}(i)^{-1} \in F \cap \left( \prod_{w \in \Omega_F} o_{F_w}^x \right) = \{ \pm 1 \}.
\]
Suppose \( N_{E/F}(\alpha) = N_{E/F}(i) = -1 \). Write
\[
i = (i_w)_w \in \prod_{w \in \Omega_F} L_w^\times.
\]
Since \( \Theta/E \) is unramified over all primes of \( E \) except \( l \) by Lemma 4.1, one has that \( \psi_{\Theta/E}(i_w) \) is trivial for all primes \( w \neq l \), where \( i_w \) is regarded as an idele whose \( w \)-component is \( i_w \) and 1 otherwise. Since
\[
N_{E/F}(i_l) = N_{E/l}(i_l) = -1,
\]
one gets
\[
\psi_{\Theta/E}(\alpha i_l) = \psi_{\Theta/E}(i_l) = \psi_{\Theta/E}(i_i) = -1 \in \mu_4
\]
by Lemma 4.2, where \( \mu_4 \) is the set of 4-th roots of unity and \( \text{Gal}(\Theta/E) \cong \mu_4 \). This contradicts to \( u \in \ker \lambda_E \).
Therefore
\[
N_{E/F}(\alpha) = N_{E/F}(i) = 1.
\]
This implies that
\[
\alpha \in R_{E/F}^1(\mathbb{G}_m)(F) \quad \text{and} \quad i \in SO_A(B).
\]
One concludes that \( \lambda_E \) is injective. The result follows from Corollary 1.13.

Finally we will give an explicit example by Proposition 4.3 such that the local–global principle is not true.

Let
\[
N_{F/\mathbb{Q}}(\delta) = 2^{s_1}7^{s_2} p_1^{e_1} \cdots p_g^{e_g} \quad \text{and} \quad D(\delta) = \{p_1, \ldots, p_g\}
\]
where \( \delta = a + b\sqrt{-14} \ (a, b \in \mathbb{Z}) \) is an integer of \( F = \mathbb{Q}(\sqrt{-14}) \). Denote \( a = 7^{s_2}a_1 \) with \( a_1 \in \mathbb{Z}, 7 \nmid a_1 \) and
\[
D_1 = \left\{ p \in D(\delta) : \left( \frac{-1}{p} \right) = \left( \frac{14}{p} \right) = 1 \text{ and } \left( \frac{7}{p} \right) = -1 \right\},
\]
\[
D_2 = \left\{ p \in D(\delta) : \left( \frac{-1}{p} \right) = -\left( \frac{14}{p} \right) = 1 \text{ and } \left( \frac{7}{p} \right) = -1 \right\},
\]
\[
D_3 = \left\{ p \in D(\delta) : \left( \frac{-1}{p} \right) = \left( \frac{14}{p} \right) = 1 \text{ and } \left( \frac{7}{p} \right) = -1 \right\}.
\]
It is clear that \( e_i \) is even for \( p_i \in D_2 \).

**Example 4.4.** Let \( F = \mathbb{Q}(\sqrt{-14}) \) and \( \delta \in o_F \) as above. Then \( \delta \) can be written as a sum of two squares over \( o_F \) if and only if

1. \( \delta \) can be written as a sum of two squares over \( o_{F_w} \) at every place \( w \) of \( F \).
2. \( D_1 \neq \emptyset \); or
\[
\left( \frac{a_1}{7} \right) = (-1) \quad \text{for } D_1 = \emptyset.
\]
Proof. Let $H$ be the Hilbert class field of $F$. By Proposition 5.31 in [3], one has that $[H : F] = 4$. Since 2 is totally ramified in $E/\mathbb{Q}$ (see Exercise 6.7 in [3]), one has $H \cap E = F$. It is clear that $K_B \supseteq H.E$. By PARI/GP computation, one gets that the class number of the order $B = o_F + o_F \sqrt{-1}$ is 4. One concludes that $K_B = H.E$.

For any $\prod_{w \in \Omega_F} (x_w, y_w) \in \prod_{w \in \Omega_F} \mathcal{X}(o_{F_w})$, we have

$$
\psi_{K_B/E} \left( \tilde{f}_E \left[ \prod_{w \in \Omega_F} (x_w, y_w) \right] \right) = \psi_{H.E/E} \left( \tilde{f}_E \left[ \prod_{w \in \Omega_F} (x_w, y_w) \right] \right)
$$

$$
= \psi_{H/F} \left( \prod_{w \in \Omega_F} N_{E/F}(\tilde{f}_E[(x_w, y_w)]) \right)
$$

$$
= \psi_{H/F}(\delta) = 1
$$

by class field theory. Therefore one only needs to consider the condition in Proposition 4.3 for $\Theta = E \left( \sqrt[4]{7} \right)$ where

$$
iGal(\Theta/E) \cong \mathbb{Z}/4\mathbb{Z} = \{\pm 1, \pm i\}
$$

and $i$ is the primitive 4-th root of unity.

If $v$ is the unique prime in $E$ above 2, then $E_v = \mathbb{Q}_2 \left( \sqrt{-1}, \sqrt{2} \right)$ and $v$ is unramified and splits into two primes in $\Theta/E$. Then

$$
\psi_{\Theta/E}(\tilde{f}_E[(x_w, y_w)]) = (-1)^{s_1}
$$

where $w = v|_F$.

If $v$ (resp. $w$) is the unique place of $E$ (resp. $F$) over 7, then $E_v = \mathbb{Q}_7 \left( \sqrt{-1}, \sqrt{7} \right)$, $F_w = \mathbb{Q}_7 \left( \sqrt{-7} \right)$ and $E_v/F_w$ is unramified. Moreover, one has

$$
\sqrt{-1} = \frac{1}{2} \left( 1 + \sqrt{-1} \right)^2 \in (E_v^\times)^2.
$$

By the local solvable condition at place $w$, one has that $s_2$ is even and

$$
\text{ord}_w(a) < \text{ord}_w \left( b \sqrt{-14} \right).
$$

Since $v$ splits into two primes in $\Theta/E$, one has the following Hilbert symbol

$$
\psi_{\Theta/E}(\tilde{f}_E[(x_w, y_w)]) = \left( \frac{x_w + y_w \sqrt{-1}}{E_v}, \sqrt{7} \right)
$$

$$
= \left( \frac{x_w + y_w \sqrt{-1}}{E_v}, \sqrt{-7} \right) = \left( \delta, \sqrt{-7} \right)
$$

$$
= \left( \frac{a, \sqrt{-7}}{F_w} \right) = \left( \frac{a, 7}{\mathbb{Q}_7} \right) = \left( \frac{a_1}{7} \right) (-1)^{s_2}
$$

by (1.5.3) Proposition and (7.1.4) Corollary in [9].

If $(p, 14N_{F/Q}(\delta)) = 1$, then all primes above $p$ are unramified in $\Theta/E$. One has

$$
\psi_{\Theta/E} \left( \tilde{f}_E \left[ \prod_{w \mid p} (x_w, y_w) \right] \right) = 1.
$$
If \( p \in D(\delta) \) with \( \left( \frac{-1}{p} \right) = -1 \), then
\[
F_w = \mathbb{Q}_p \left( \sqrt{-1} \right) = \mathbb{Q}_p \left( \sqrt{-14} \right)
\]
by the local solvability at \( w \) with \( w|p \). Therefore \( \sqrt{7} \in F_w \). The Hilbert symbol
\[
\left( \frac{p, \sqrt{7}}{F_w} \right) = \left( \frac{p, (-1)^t7}{\mathbb{Q}_p} \right) = 1
\]
by (1.5.3) Proposition and (7.1.4) Corollary in [9], where
\[
t = \begin{cases} 
0 & \text{if } \left( \frac{7}{p} \right) = 1 \\
1 & \text{otherwise.}
\end{cases}
\]
This implies that \( \sqrt{7} \in (F_w^\times)^2 \) by 63:12 in [11]. One concludes that each prime above \( w \) splits completely in \( \Theta/E \). Hence
\[
\psi_{\Theta/E}(\tilde{f}_E[(x_w, y_w)]) = 1.
\]
We restrict ourselves to \( p \in D(\delta) \) with \( \left( \frac{-1}{p} \right) = 1 \).

If one further has \( \left( \frac{7}{p} \right) = 1 \) and \( x^4 \equiv 7 \mod p \) is solvable, then all primes above \( p \) split completely in \( \Theta/E \). Hence
\[
\psi_{\Theta/E}(\tilde{f}_E[(x_w, y_w)]) = 1
\]
for \( w|p \).

If \( \left( \frac{7}{p} \right) = 1 \) and \( x^4 \equiv 7 \mod p \) is not solvable, then
\[
\psi_{\Theta/E}(\tilde{f}_E[(x_w, y_w)]) = \begin{cases} 
\left( \frac{\delta, \sqrt{7}}{F_w} \right) = (-1)^e & \text{if } p \in D_3 \setminus D_1 \\
1 & \text{otherwise}
\end{cases}
\]
where \( e = \text{ord}_w(\delta) \) with \( w|p \).

If \( p \in D_2 \), then \( p \) is inert in \( F/\mathbb{Q} \) and splits completely into \( v_1 \) and \( v_2 \) in \( E \). Moreover, each \( v_i \) with \( i = 1, 2 \) splits into two primes in \( \Theta \). Therefore
\[
\psi_{\Theta/E}(\tilde{f}_E[(x_w, y_w)]) = \left( \frac{\delta, \sqrt{7}}{F_w} \right) = (-1)^e
\]
with \( w|p \) and \( e = \text{ord}_p(N_{F/\mathbb{Q}}(\delta)) \).

If \( p \in D_1 \), then \( p \) splits completely in \( F/\mathbb{Q} \). Fix \( w|p \) in \( F \). One has that \( w \) splits into \( v_1 \) and \( v_2 \) completely in \( E/F \). Let \( i \) be the image of the Frobenius of \( v_1 \) inside \( \text{Gal}(\Theta/E) \). Then the image of the Frobenius of \( v_2 \) inside \( \text{Gal}(\Theta/E) \) is \(-i\) by considering the action of \( \text{Gal}(E/F) \). This implies that
\[
\psi_{\Theta/E}(\tilde{f}_E[(x_w, y_w)]) = i^a \cdot (-i)^b = (-1)^b \cdot i^e
\]
where \( a = \text{ord}_w(x_w \pm y_w \sqrt{-1}) \) and \( b = \text{ord}_w(x_w \mp y_w \sqrt{-1}) \) and \( e = a + b = \text{ord}_w(\delta) \).
Now we summarize the above computation as follows. For any $p_i \in D(\delta)$, one has

$$
\psi_{\Theta/E} \left( \prod_{w|p_i} \tilde{f}_E[(x_w, y_w)] \right) = \begin{cases} 
(-1)^{b_i} \cdot i^{e_i} & \text{if } p_i \in D_1 \\
(-1)^{e_i} & \text{if } p_i \in D_2 \\
1 & \text{if } p_i \in D_3 \setminus D_1 
\end{cases}
$$

with $\prod_{w|p_i} (x_w, y_w) \in \prod_{w|p_i} X(o_{F_w})$ and

$$
b_i = \sum_{w|p_i} \text{ord}_w \left( x_w \mp y_w \sqrt{-1} \right).
$$

Since $\delta \in N_{E_v/F_w}(E_v^\times)$ for $v|w|2$, one has

$$
1 = \left( -1, \frac{N_{F_w/Q_2}(\delta)}{Q_2} \right) = \left( \frac{7, n'}{Q_2} \right)
$$

where $n' = p_1^{e_1} \cdots p_s^{e_s}$. Since $E_v = F_w \left( \sqrt{7} \right)$ and $\delta \in N_{E_v/F_w}(E_v^\times)$ for $v|w|7$, one has

$$
1 = \left( \frac{7, N_{F_w/Q_2}(\delta)}{Q_7} \right) = \left( \frac{7, n'}{Q_7} \right).
$$

By the Hilbert reciprocity law (see 71:18 in [11]), one obtains

$$
\prod_{i=1}^g \left( \frac{7}{p_i} \right)^{e_i} = 1.
$$

If $\left( \frac{-1}{p_i} \right) = -1$ or $\left( \frac{-14}{p_i} \right) = -1$ for $p_i \in D(\delta)$, then $e_i$ is even by the local conditions. Therefore we have

$$
\sum_{p_i \in D_1} e_i \equiv 0 \mod 2
$$

and $\psi_{\Theta/E}$ takes value $\{\pm 1\}$ over $\prod_{w \in \Omega_F} X(o_{F_w})$.

When $D_1 \neq \emptyset$, then $\psi_{\Theta/E} (\tilde{f}_E[(x_w, y_w)])$ takes two different values by proper choice of the local solution $(x_w, y_w) \in X(o_{F_w})$ for $w|p_i$ with $p_i \in D_1$. Hence there exists

$$
\prod_{w \in \Omega_F} (x_w, y_w) \in \prod_{w \in \Omega_F} X(o_{F_w}) \quad \text{such that} \quad \psi_{\Theta/E} \left( \prod_{w \in \Omega_F} \tilde{f}_E[(x_w, y_w)] \right) = 1.
$$

When $D_1 = \emptyset$, one concludes that $X(o_F) \neq \emptyset$ if and only if

$$
\left( \frac{a_1}{7} \right) = (-1)
$$

by Proposition 4.3. \hfill \Box

**Acknowledgments**

The authors would like to thank Colliot-Thélène for helpful comments on the early version of the paper. The first author is supported by NSFC, grant # 10671104 and grant DE 1646/2-1 of the
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