Validation of the Cooray-Rubinstein (C-R) formula for a rough ground surface by using three-dimensional (3-D) FDTD

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[1] In this paper, we have extended the Cooray-Rubinstein (C-R) approximate formula into the fractal rough ground surface and then validate its accuracy by using three-dimensional (3-D) finite-difference time-domain (FDTD) method at distances of 50 m and 100 m from the lightning channel. The results show that the extended C-R formula has an accepted accuracy for predicting the lightning-radiated horizontal electric field above the fractal rough and conducting ground, and its accuracy increases a little better with the higher of the earth conductivity. For instance, when the conductivity of the rough ground is 0.1 S/m, the error of the peak value predicted by the extended C-R formula is less than about 2.3%, while its error is less than about 6.7% for the conductivity of 0.01 S/m. The rough ground has much effect on the lightning horizontal field, and the initial peak value of the horizontal field obviously decreases with the increase of the root-mean-square height of the rough ground at early times (within several microseconds of the beginning of return stroke).

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1. Introduction

[2] The accurate estimation of the lightning-induced voltages on overhead line is very important for the lightning protection. According to the field-line coupling model [Agrawal et al., 1980], the total induced wave on the overhead line is composed of the two components named by the incident- and scatter-induced waves. The incident- and scatter-induced waves are caused by the vertical and horizontal electric field, respectively; however, it is noted that the dominance of any of these field components depends on the tilt of the lightning return stroke channel and the orientation of the overhead line with respect to the lightning channel. Within hundreds of meters from the lightning channel, the vertical electric field is less affected by the finite conducting ground; however, the horizontal electric field is very sensitive to the finite conductivity of the earth. There are several approximate formulas for predicting the lightning horizontal electric field in frequency domain and time domain. [Barbosa and Paulino, 2007; Barbosa et al., 2013; Caligaris et al., 2008; Cooray, 1992, 2002; Delfino et al., 2008; Khosravi et al., 2013; Norton, 1936; Rachidi et al., 1996; Rubinstein, 1996; Wait, 1997; Zeddam and Degauque, 1987]. Among them, the most remarkable one is the Cooray-Rubinstein (C-R) approximate formula in frequency domain [Cooray, 1992, 2002; Rubinstein, 1996], which has been proved to have a reasonably good accuracy at distances of tens of meters to 1 km with a conductivity ranging from 0.1 S/m to 0.001 S/m for homogeneously conducting ground [Caligaris et al., 2008; Cooray, 2010; Shoory et al., 2005]. In the last two years, the Cooray-Rubinstein (C-R) formula has been extended into the horizontally stratified ground and mixed propagation path. For instance, Shoory et al. [2011] have extended the C-R formula into the horizontally stratified conducting ground, and Zhang et al. [2012] extended it into the mixed path and estimated its accuracy at distances of 100 m to 1000 m from the lightning channel by using finite-difference time-domain (FDTD) method.

[3] Recently, Zhang et al. [2013] have further extended the C-R formula into a rough and ocean land mixed propagation surface and analyzed the propagation effect of the roughness of the ocean surface and land section on the lightning-radiated horizontal field. However, the accuracy of the extended C-R formula over a rough ground surface with homogeneous or vertically stratified (mixed) conductivity has not been validated by using other techniques, which will restrict the extensive applicability of the C-R formula. More recently, Li et al. [2013] have developed a three-dimensional (3-D) FDTD technique for simulating the lightning field over the two-dimensional (2-D) rough boundary condition. It is noted that the effect of the 2-D surface roughness on the horizontal field cannot be ignored even at a distance of 100 m from the lightning channel, and the increase of the land roughness results in a lower field magnitude because of the more propagation attenuation, compared with smooth ground surface.
However, although the 3-D FDTD method can simulate the lightning field over the rough and conducting ground, the FDTD method is very complex and time consuming; in practical engineering applications, the approximate method maybe more efficient and valuable than the FDTD method. Therefore, in the following section, we will briefly extend C-R formula into a rough ground with the fractal geometry, and then validate its accuracy by using our 3-D FDTD method proposed by Li et al. [2013].

2. Extension of the C-R Formula Considering the Rough Ground Surface

Based on the C-R approximation for homogenous and finitely conducting ground [Cooray, 2010], the C-R formula considering the rough and finitely conducting ground can be expressed as below:

\[ E_{h,a}(z,d,jo) = -H_{phi}(0,d,jo) \cdot W(0,d,jo) Z + E_{r,a}(z,d,jo), \]  

where \(E_{h,a}(z,d,jo)\) and \(E_{r,a}(z,d,jo)\) are the lightning horizontal electric fields above the finitely and perfectly conducting ground. \(W(0,d,jo)\) is the attenuation function, and \(Z\) is the effective surface impedance of the rough propagation path given by Cooray [2010].

\[ W(0,d,jo) = 1 - j \sqrt{\pi p} \exp(-p) \text{erfc}(j \sqrt{p}), \]  

\[ p = -\frac{j \omega_0 d^2}{2c} \Delta^2, \]  

\[ \Delta = \sqrt{\frac{\epsilon_0}{\mu_0}} Z, \]  

The normalized surface impedance is \(\Delta\) corresponding to the rough ground. For the rough ground surface, the normalized surface impedance consists of two terms: one is the impedance of the lower medium when the surface is perfectly smooth and the other accounting for roughness [Barrick, 1971a, 1971b].

\[ \Delta = \Delta^0 + \Delta', \]  

\[ \Delta^0 = \frac{(j \omega_0 \sigma + j \omega_0 (\epsilon_0 - 1))^1/2}{\sigma + j \omega_0 \epsilon_0}, \]  

\[ \Delta' = \frac{1}{4} \int_{-\infty}^{+\infty} G(\gamma, \eta) V(\gamma, \eta) d\eta. \]

\[ G(\gamma, \eta) = \frac{\gamma^2 + b \Delta(\gamma^2 + \eta^2 - \omega \eta/c)}{b + \Delta^0 (b^2 + 1)} + \Delta^0 \left( \frac{\gamma^2 - \eta^2}{2} + \omega \eta/c \right). \]

\[ b = \frac{c}{\omega} \left( \frac{\omega \sigma}{c} - \left( r + \omega \sigma \right)^2 - \gamma \right)^{1/2}. \]

where \(\Delta^0\) is the normalized surface impedance of the smooth surface and \(\Delta'\) is the increment of the normalized surface impedance due to the roughness given by Barrick [1971a, 1971b]. The height density spectral is \(V(\gamma, \eta)\) corresponding to the irregular terrain, \(\gamma\) and \(\eta\) are the radial wave numbers (or spatial frequencies) along the \(x\) and \(y\) directions. The dielectric constant and magnetic permeability of free space are \(\epsilon_0\) and \(\mu_0\), respectively. From equations 1–7, we can see that the height density spectral is crucial for simulating the field propagation along the irregular terrain.

3. Simulation of the Two-Dimensional (2-D) Rough Ground Surface

Since the natural ground surface is generally neither purely random nor purely periodic and often anisotropic, a normalized two-dimensional (2-D) band limited Weierstrass fractal function is employed to simulate the rough land surface as below [Ren and Guo, 2008]:

\[ \text{Figure 1.} \quad \text{The 2-D rough surface simulated by using a normalized 2-D band limited Weierstrass fractal function, and the triangle shows the position of the lightning striking point.} \]
where \( \delta \) is the root-mean-square height (RMSH) of the rough surface, \( D \) \((2 < D < 3)\) is the fractal dimension, \( K \) is the fundamental wave number, \( b \) is the fundamental spatial frequency, \( N \) and \( M \) are the numbers of harmonics, and \( \phi_{nm} \) is a phase term that has a uniform distribution over the interval \([-\pi, \pi)\). In Figure 1, the 2-D rough surface is simulated by using the normalized 2-D band limited Weierstrass fractal function in equation 8 with \( N_1 = 0, N_2 = 6, M = 7, b = 1.6, D = 2.3, \delta = 2, \) and \( \kappa = 100 \).

4. Calculation of the Height Density Spectral for the Rough Ground

[8] Generally, there are many statistics parameters (e.g., root-mean-square height autocorrelation function and height density spectral) depicting the rough characteristics of an irregular terrain. RMSH expression is given by [Guo et al., 2009]

\[
\delta = \sqrt{E[f^2(x,y)] - [E[f(x,y)]]^2}
\]

[10] where \( f(x,y) \) is a randomly rough surface function, \( E[f(x,y)] \) and \( E[f^2(x,y)] \) are the mathematical expectation values of the functions \( f(x,y) \) and \( f^2(x,y) \), respectively. For a certain randomly irregular terrain, the autocorrelation function is expressed as below [Guo et al., 2009].

\[
R(x,y) = E[f(x,y)f(x+\Delta x,y+\Delta y)]
\]

[11] where \((\Delta x,\Delta y)\) represents the increment from one point \((x,y)\) to another randomly \(((x+\Delta x,y+\Delta y))\). According to Wiener-Khinchin theory [Cohen, 1998], the height density spectral is the two-dimensional Fourier fast transform of the autocorrelation function.

\[
V(\gamma,\eta) = \int_{-\infty}^{+\infty} R(x,y) \exp[-j2\pi(\gamma x + \eta y)] dx dy
\]

[12] The equation 7 has been proved to have suitable for deterministic periodic surfaces as well as random rough surfaces [Barrick, 1971a, 1971b]. From equations 8–11, the height density spectral \( V(\gamma,\eta) \) of the simulated rough ground surface in Figure 1 can be obtained, as shown in Figure 2.

5. Validation of the C-R Formula by Using 3-D FDTD

[13] The 3-D FDTD model used in this paper is presented in Figure 3, the working space is \( 211 \, \text{m} \times 51 \, \text{m} \times 3501 \, \text{m} \) which is divided into square cells of \( \Delta x \times \Delta y \times \Delta z = 1 \, \text{m} \times 1 \, \text{m} \times 1 \, \text{m} \), the time increment is set to 1.66 ns. The air and ground are both split by Yee’s grid units and 15 planes of the perfectly matching layer is adopted as the absorption boundary condition [Berenger, 1996; Yee, 1966]. On the interface between the air and ground the electric and magnetic parameters are taken as the linear average of both mediums [Li et al., 2013].

[14] The lightning channel is represented by a vertical array of current sources [Baba and Rakov, 2003]. Each current source has a length of 1 m. Figure 3 shows a vertical phased
Figure 4. Lightning horizontal electric field at a height of $h = 10$ m above the rough land with different RMSH at distances of 50 m and 100 m from the lightning channel, and the conductivity of the ground is 0.1 S/m.

Figure 5. Similar to Figure 4 but for the rough ground with the conductivity of 0.01 S/m.
array of 3500 current sources on a perfectly conducting plane. The influence of reflections from the upper end of the 3500 m long channel does not appear in calculated waveforms of lightning horizontal electric field within the first 4.0 \( \mu s \) in this paper. The channel base current is representative of a subsequent stroke and has 12 kA peak and maximum rate of rise of 40 kA, as proposed by Rachidi et al. [2001], and the modified transmission line model with a linear decay of current with height is adopted [Rakov and Dulzon, 1991].

\[
I(z, t) = (1 - z/H)I(0, t - z/v)
\]  

(11)

where the height of the channel \( H \) is assumed to be 7.5 km, the return stroke speed is \( v = 1.5 \times 10^8 \) m/s. Figures 4 and 5 show the simulated lightning horizontal electric field at a height of \( h = 10 \) m above the rough land with different RMSH at distances of 50 m and 100 m from the lightning channel, and the conductivity of the ground is 0.1 S/m and 0.01 S/m. On one side, note that, with the increase of the RMSH, the initial horizontal field decreases obviously due to the field attenuation. When RMSH = 5 m, the field peak value over the rough ground surface is nearly the half of that over the smooth ground. The effect of the roughness on the lightning horizontal field cannot be ignored. On the other side, we can see that the extended C-R formula has an accepted accuracy for predicting the horizontal field over the rough ground surface, and its accuracy is a little better for the higher conductivity than that for the lower conductivity. Because the surface impedance in the extended C-R formula is calculated according to the Barrick theory [Barrick, 1971a, 1971b], the roughness of the ground surface can be expressed as an increase in the normalized surface impedance of the smooth ground. As is well known, the Barrick theory has a better accuracy when the medium below surface is highly conducting; therefore, our extended C-R formula in this paper has a little better accuracy for higher conductivity.

### References


### Table 1. Peak and Rise Time of the Lightning Horizontal Electric Field Using the C-R Formula and 3-D FDTD Simulations

<table>
<thead>
<tr>
<th>d (m)</th>
<th>Conductivity (S/m)</th>
<th>RMSH (m)</th>
<th>Peak (V/m)</th>
<th>Rise time (( \mu s ))</th>
</tr>
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<tbody>
<tr>
<td>50</td>
<td>0.1</td>
<td></td>
<td>C-R Formula</td>
<td>FDTD</td>
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<tr>
<td>2</td>
<td>4388.22</td>
<td>4397.74</td>
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