Flow of Homeomorphisms and Stochastic Transport Equations

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Abstract: We consider Stratonovich stochastic differential equations with drift coefficient $A_0$ satisfying only the condition of continuity

$$|A_0(x) - A_0(y)| \leq C|x - y|r(|x - y|^2)$$

for all $|x - y| \leq c_0$, where $r$ is a positive $C^1$ function defined on a neighborhood $[0, c_0]$ of 0 such that $\int_0^{c_0} \frac{ds}{s^{1/3}} = +\infty$ (Osgood condition), and $s \to r(s)$ is decreasing while $s \to sr(s^2)$ is increasing. We prove that the equation defines a flow of homeomorphisms if the diffusion coefficients $A_1, \ldots, A_N$ are in $C^{1+\delta}(\mathbb{R}^d, \mathbb{R}^d)$. If $r(s) = (\log \frac{1}{s}) \cdots (\log \frac{1}{s^k})$, we prove limit theorems for Wong-Zakai approximation as well as for regularizing the drift $A_0$. As an application, we solve a class of stochastic transport equations.

Keywords: Flow of homeomorphisms; Non-Lipschitz condition; Stochastic differential equation; Stochastic transport equation; Wong-Zakai approximation.

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1. INTRODUCTION

Let \( v = v(t, x) \) be a vector field on \( \mathbb{R}^d \). It is well-known that if \( v \) is smooth, the following transport equation in the study of turbulence

\[
\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad t > 0, \quad \theta(0, x) = \theta_0(x),
\]

(1.1)

admits the unique solution given by \( \theta(t, x) = \theta_0(X_{s,t}(x)) \), where \( X_{s,t}(x) \) is the flow of diffeomorphisms generated by the vector field \( v 
\)

\[
dX_{s,t} = v(t, X_{s,t}) dt, \quad t > s, \quad X_{s,s} = x.
\]

(1.2)

When \( v \) is a continuous vector field satisfying the Osgood condition, the differential equation (1.2) gives rise to a flow of homeomorphisms of \( \mathbb{R}^d \) and \( \theta(t, x) = \theta_0(X_{s,t}(x)) \) solves (1.1) in distribution sense.

One motivation of this work is to consider the transport equation when the velocity field \( v \) is given by

\[
v(t, x) = \sum_{k=1}^N A_k(x) \dot{w}_k^i + A_0(x),
\]

where \( A_0, A_1, \ldots, A_N \) are vector fields on \( \mathbb{R}^d \) and \( \dot{w}_i \) is the Gaussian white noise in \( \mathbb{R}^N \). Such an equation will be studied in Section 5.

Consider the following Stratonovich stochastic differential equation (SDE)

\[
dX_t = \sum_{i=1}^N A_i(X_t) \circ d\dot{w}_i + A_0(X_t) dt, \quad X_0 = x,
\]

(1.3)

where \( w_t = (w^1_t, \ldots, w^N_t) \) is a standard Brownian motion on \( \mathbb{R}^N \). It is well known that if the coefficients \( A_1, \ldots, A_N \) are in \( C^2 \), bounded with their bounded first and second order derivatives and \( A_0 \) is in \( C^1 \) bounded with its bounded first order derivative, the limit theorem for (1.3) holds. More precisely, we regularize the Brownian motion paths as

\[
\dot{w}_i(t) = 2^n(w_{i+1,2^{-n}} - w_{i,2^{-n}}), \quad t \in [i2^{-n}, (i + 1)2^{-n}[, \quad i \in \mathbb{Z},
\]

and consider the ordinary differential equations

\[
dX^n_t = \left[ \sum_{i=1}^N A_i(X^n_t) \dot{w}_i(t) + A_0(X^n_t) \right] dt, \quad X^n_0 = x;
\]

(1.4)

then (see [16]) almost surely, \( X^n_t(x, w) \) converges to \( X_t(x, w) \) uniformly with respect to \( (t, x) \) in any compact subset of \( [0, +\infty[ \times \mathbb{R}^d \). As an
application of this limit theorem, the SDE (1.3) defines a flow of
homeomorphisms and the inverse of the flow can be given explicitly,
contrary to Kunita’s flow of homeomorphisms of SDE satisfying global
Lipschitz conditions. More precisely, Fix $T > 0$ and consider
\[ \hat{w}_t^T = w(T) - w(T - t). \]

Let $\hat{X}_t^T(x, \hat{w}^T)$ be the solution to the SDE
\[ d\hat{X}_t^T = \sum_{i=1}^N A_i(\hat{X}_t^T) \circ d(\hat{w}_t^T' - A_0(\hat{X}_t^T)) dt, \quad \hat{X}_0^T = x. \]

Then using the limit theorem, it holds true that
\[ X_{T-t}(x, w) = \hat{X}_t^T(x, \hat{w}^T), \]
\[ \hat{X}_t^T_t(x, \hat{w}^T) = X_t(x, \hat{w}^T), \quad 0 \leq t \leq T. \]

Therefore, the inverse map $x \mapsto X_{T-1}(x, w)$ is given explicitly by $x \mapsto \hat{X}_{T}^T(x, \hat{w}^T)$.

Recently in [4], the authors relaxed the condition on the
boundedness of derivatives to get a flow of homeomorphisms (see a
previous work [12] for slightly more regular coefficients); to this end, a
limit theorem with modified coefficients was established.

On the other hand, studies for the non-Lipschitz Itô SDE
\[ dx_t = \sum_{i=1}^N A_i(x_t) dw_t^i + \tilde{A}_0(x_t) dt, \quad x_0 = x \tag{1.5} \]
have risen a great interest recently (see [5, 7, 9, 11, 15, 19, 21, 22]).
Namely, suppose that for $|x - y| \leq c_0$,\[
\sum_{i=1}^N |A_i(x) - A_i(y)|^2 \leq C |x - y|^2 r(|x - y|^2), \tag{1.6}
\]
\[ |
\tilde{A}_0(x) - \tilde{A}_0(y)| \leq C |x - y| r(|x - y|^2) \tag{1.7}
\]
where $r$ is a strictly positive, $C^1$-function defined on a neighborhood
$[0, c_0]$ of 0 satisfying $\int_0^{\infty} \frac{dt}{r(t)} = +\infty$, then the Itô SDE (1.5) has pathwise
uniqueness and the non contact property [5]. If moreover $r(s) = \log \frac{1}{s}$
and the SDE (1.5) is strictly conservative, then the dependence with
respect to initial values is continuous [5]. Under a slightly stronger
condition, (1.5) defines a flow of homeomorphisms (see [21]). We
emphasize that if the function $r(s) = (\log \frac{1}{s})(\log \log \frac{1}{s})$, then (see [5])
\[ E(|x_t(x) - x_t(y)|^p) \leq C_p \exp\left( -\left( \log \frac{1}{|x - y|^p} \right)^{\rho} \right), \]
which is not bounded by \(|x - y|^2\) with some \(x > 0\); therefore the Kolmogoroff theorem fails to be applied to get a continuous modification. However, for the ordinary differential equation \(\frac{dx}{dt} = A_0(x), x_0 = x\), if \(A_0\) satisfies the Osgood condition, then \(x \to x_t(x)\) is a homeomorphism of \(\mathbb{R}^d\) [5]. The natural question arises: is it possible to get a stochastic flow of homeomorphisms for coefficients with modulus of continuity beyond \(s \log \frac{1}{s}\)?

In the case where the diffusion coefficient is non-degenerated with very singular drift \(A_0\), the strong solutions of (1.5) have been studied by Krylov and Röckner in [9], and later by Zhang in [22]. In the same spirit, a result concerning flow of homeomorphisms was obtained by Zhang [21].

The main purpose of this work is to remove the condition of non-degeneracy, the price to pay is that the diffusion coefficients are assumed to be sufficiently regular. Let’s now explain the main results in this article.

**Theorem 1.** Let \(A_1, \ldots, A_N\) be vector fields in \(C^{3+\delta}\), bounded with bounded derivatives up to order 3, \(A_0\) be a continuous vector field satisfying

(i) \(|A_0(x)| \leq C_1(1 + |x|^{1-\epsilon_0}), x \in \mathbb{R}^d\),

(ii) \(|A_0(x) - A_0(y)| \leq C_2|x - y|^{\epsilon}(|x - y|^2)\), for all \(|x - y| \leq c_0\) where \(C_1, C_2 > 0, \epsilon_0 \in [0, 1], c_0 \in [0, 1]\) are constants, \(r : [0, c_0] \to [1, +\infty]\) is a \(C^1\)-function defined on a neighborhood of the origin 0, such that

(iii) \(\int_0^{c_0} \frac{\alpha(t)}{s(t)} = +\infty, \lim_{s \to 0} \frac{\alpha(s)}{s} = 0\),

(iv) \(s \to r(s)\) is decreasing and \(s \to sr(s^2)\) is increasing on \([0, c_0]\).

Then almost surely, for each \(t > 0\), \(x \to X_t(x, u)\) is a homeomorphism of \(\mathbb{R}^d\).

This theorem will be proved in Section 2. Note that the modulus of continuity of the coefficients in [21] are \(s \log \frac{1}{s}\), while here \(A_0\) satisfies the general Osgood condition. Note also that if the diffusion coefficients are equal to zero, the random vector field \(\tilde{A}_0(t, x)\) defined in (2.5) is equal to \(A_0\); therefore the situation is reduced to the ordinary one.

When we deal with the Stratonovich SDE, it is natural to ask whether the Wong–Zakai approximation does work, that is, the solutions of ordinary differential equations (1.4) convergent to solutions of SDE (1.3) in some sense. Let \(k \geq 0\), we denote \(\log_{k} t = t, \log_{k} t = \log \log_{k+1} (t)\). Then we have

**Theorem 2.** Suppose furthermore that \(r(s) = (\log \frac{1}{s}) \cdots (\log \frac{1}{s})\), then for any \(p > 1\) and \(R > 0\),

\[
\lim_{n \to +\infty} E\left( \sup_{0 \leq t \leq 1} \sup_{|x| \leq R} |X^n_t(x) - X_t(x)|^p \right) = 0.
\]
This result is useful in order to get some convergence on the inverse of the flow. Note that the usual machinery developed in [1] or in [8] does not work for proving Theorem 2. In fact, for the estimate of moments for two point motion \((X^n)(x), X^n(y))\), the condition (ii) for \(A_0\) yields to

\[
|X^n_t(x) - X^n_t(y)| \leq |X^n_n(x) - X^n_n(y)| + L \sum_{i=1}^N |\dot{u}_i(s_n)| \int_{s_n}^t |X^n_t(x) - X^n_t(y)| du
\]

\[
+ C_2 \int_{s_n}^t |X^n_t(x) - X^n_t(y)| r(|X^n_n(x) - X^n_n(y)|^2) du,
\]

where \(s_n = [2^n s, 2^n s + 2^{-n}]\) and \(\sigma \in [s_n, s_n^+]\). Even for \(r(s) = \log(1/s)\), the above inequality implies that

\[
|X^n_t(x) - X^n_t(y)| \leq (|X^n_n(x) - X^n_n(y)|) e^{-C_2 s} e^{-1} \sum_{i=1}^N |\dot{u}_i(s_n)|.
\]

The exponent \(e^{-C_2 s} \leq 1\) invalidates the method of moment estimates.

We will use the method of decomposition (see Ocône-Pardoux [17]), the proof of Theorem 2 will be given in Section 3.

Since the theory for stochastic flows is well established for regular coefficients (see [1, 3, 8, 14, 18]), it is natural to regularize the drift \(A_0\). This will be done in Section 4.

Finally in Section 5, using the convergence results in Sections 3 and 4, we solve the stochastic transport equation (see Theorem 5.1)

\[
d\theta(t, x) + \sum_{i=1}^N \langle \nabla \theta(t, x), A_i(x) \rangle \circ dw^i_t + \langle \nabla \theta(t, x), A_0(x) \rangle dt = 0.
\]

### 2. FLOW OF HOMEOMORPHISMS

Let’s recall first a result concerning ordinary differential equations.

**Theorem 2.1.** Let \(v : [0, +\infty[ \times \mathbb{R}^d \to \mathbb{R}^d\) be a time dependent vector field. Consider

\[
\frac{dy}{dt} = v(t, y), \quad y_0 = x.
\]

Suppose that for each \(T > 0\), there exists a constant \(C_T > 0\) such that

\[
|v(t, x)| \leq C_T (1 + |x|), \quad (t, x) \in [0, T] \times \mathbb{R}^d,
\]

and

\[
|v(t, x) - v(t, y)| \leq C_T |x - y| r_1(|x - y|^2) \quad \text{for all} \ t \in [0, T], \ |x - y| < \delta_0,
\]

where \(r_1 : [0, \delta_0] \to [1, +\infty[\) is a continuous function such that \(\int_0^{\delta_0} \frac{dr}{r^{1/2}} = +\infty\). Then for each \(t \geq 0\), \(x \to y_t(x)\) is a homeomorphism of \(\mathbb{R}^d\).
Proof. The growth condition (2.2) insures that the solution of (2.1) does not explode at a finite time. The condition (2.3) allows to prove that the differential equation (2.1), admits a unique solution \( y_t(x) \) and \( (t, x) \to y_t(x) \) is continuous (for a detailed proof, see [5; Section 2]). Fix a \( T > 0 \) and consider

\[
\frac{dz_t}{dt} = -v(T - t, z_t), \quad z_0 = x. \tag{2.4}
\]

By uniqueness of solutions, we have

\[
y_{T} - t(x) = \tilde{z}_T(x), \quad z_T(x) = y_T(x).
\]

Let \( t = T \), we see that

\[
y_{T}^{-1} - t(x) = \tilde{z}_T(x).
\]

□

Now consider the following Stratonovich SDE without drift:

\[
d\tilde{X}_t = \sum_{i=1}^{N} A_i(\tilde{X}_t) \circ dw^i_t, \quad \tilde{X}_0 = x.
\]

It is well-known that under the conditions on \( A_1, \ldots, A_N \) in Theorem 1, the solutions of the above SDE admit a version \( \tilde{X}_t(x, w) \) such that there exists a full subset \( \Omega_0 \), for each \( w \in \Omega_0 \) and each \( t > 0, x \to \tilde{X}_t(x, w) \) is a \( C^2 \)-diffeomorphism of \( \mathbb{R}^d \). Set \( \varphi_t(x) = \tilde{X}_t(x, w) \). Let \( J_t(x) = (\partial_{x} \varphi_t(x)) \) be the Jacobian matrix of \( \varphi_t(\cdot) : \mathbb{R}^d \to \mathbb{R}^d \) and \( K_t(x) = (J_t(x))^{-1} \) the inverse of \( J_t(x) \). Define for \( w \in \Omega_0 \),

\[
\tilde{A}_0(t, x) = K_t(x) A_0(\varphi_t(x)). \tag{2.5}
\]

We consider the differential equation

\[
dY_t = \tilde{A}_0(t, Y_t) dt, \quad Y_0 = x. \tag{2.6}
\]

Then the solutions of (1.3) can be expressed (see [17]) as

\[
X_t(x) = \varphi_t(Y_t(x)). \tag{2.7}
\]

Lemma 2.2. Under the conditions in Theorem 1, for \( T > 0 \), there exist \( \varepsilon_1 > 0 \) and \( \Phi_T \in \cap_{p=1}^\infty L^p(\Omega) \) such that

\[
|\tilde{A}_0(t, x)| \leq \Phi_T(1 + |x|^{1-\varepsilon_1}), \quad (t, x) \in [0, T] \times \mathbb{R}^d. \tag{2.8}
\]

Proof. Choose \( a > 1 \) such that \( a(1 - \varepsilon_0) < 1 \). Take \( \varepsilon > 0 \) small enough such that \( b := 1 - \varepsilon - a(1 - \varepsilon_0) > 0 \). It is known that there exist two functions \( F, G \geq 1 \) in all \( L^p(\Omega) \) such that for all \( (t, x) \in [0, T] \times \mathbb{R}^d \),

\[
|\varphi_t(x)| \leq F \cdot (1 + |x|^b), \quad \|K_t(x)\| \leq G \cdot (1 + |x|^b) \tag{2.9}
\]

where \( \|\cdot\| \) denotes the norm of matrices. By sub linear growth on \( A_0 \),

\[
|A_0(\varphi_t(x))| \leq C_1(1 + |\varphi_t(x)|^{1-\varepsilon_0}) \leq F_1(1 + |x|^{b(1-\varepsilon_0)})
\]
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where $F_1 = 2C_1F^{1-\epsilon_0}$. Therefore,

$$\left|\tilde{A}_0(t, x)\right| \leq \left\|K_t(x)\right\| \cdot |A_0(\varphi_t(x))| \leq G(1 + |x|^\delta) \cdot F_1(1 + |x|^{\delta(1-\epsilon_0)})$$

$$= GF_1(1 + |x|^{\delta} + |x|^{\delta(1-\epsilon_0)} + |x|^{1-\epsilon}) \leq 3GF_1(1 + |x|^{1-\epsilon})$$

where $\epsilon > 0$ is small enough. We get (2.8).

The vector field $\tilde{A}_0$ does not satisfy the condition (2.3). For this reason, we will introduce truncated functions which make Theorem 2.1 applicable. Let $\ell > 0$, define $\beta_\ell \in C_0^\infty(\mathbb{R}^d, [0, 1])$ such that

$$\beta_\ell(x) = \begin{cases} 1, & \text{if } |x| \leq \ell; \\ 0, & \text{if } |x| > \ell + 1. \end{cases}$$

Let

$$\tilde{A}_0'(t, x) = \beta_\ell(x)K_t(x)A_0(\varphi_t(x)). \quad (2.10)$$

**Lemma 2.3.** For $T > 0$ and $\ell \geq 1$ given, there exist $\delta_{T, \ell} > 0$ and a constant $C_{T, \ell} > 0$ such that

$$\left|\tilde{A}_0'(t, x) - \tilde{A}_0'(t, y)\right| \leq C_{T, \ell}|x - y|r(|x - y|^2),$$

if $t \in [0, T]$, $|x - y| \leq \delta_{T, \ell}$. \quad (2.11)

**Proof.** Let

$$\alpha_{T, \ell} = \left(\sup_{t \leq T} \sup_{|x| \leq \ell+1} \|J_t(x)\|\right) \lor 1. \quad (2.12)$$

If $|x| \lor |y| \leq \ell + 1$. By definition of $\alpha_{T, \ell}$, we have

$$|\varphi_t(x) - \varphi_t(y)| \leq \alpha_{T, \ell}|x - y|.$$ 

Let

$$\delta_{T, \ell} = \epsilon_0/\alpha_{T, \ell}. \quad (2.13)$$

Then for $|x - y| \leq \delta_{T, \ell}$, the condition (ii) and (iv) in Theorem 1 give

$$|A_0(\varphi_t(x)) - A_0(\varphi_t(y))| \leq C_2\alpha_{T, \ell}|x - y|r(\alpha_{T, \ell}^2|x - y|^2)$$

$$\leq C_2\alpha_{T, \ell}|x - y|r(|x - y|^2). \quad (2.14)$$

Set

$$\tilde{\alpha}_{T, \ell} = \|\beta_\ell\|_\infty \cdot \left(\sup_{t \leq T} \sup_{|x| \leq \ell+1} \|K_t(x)\|\right) + \sup_{t \leq T} \sup_{|x| \leq \ell+1} \|\partial_t K_t\|.$$ \quad (2.15)
where \( \| \cdot \|_\infty \) denotes the uniform norm over \( \mathbb{R}^d \). Then
\[
\| \beta_i(x)K_i(x) - \beta_i(y)K_i(y) \| \leq \tilde{a}_{T,\ell}|x - y|.
\]

Let
\[
C_{T,\ell} = C_2 \left( \sup_{t \leq T} \sup_{|\xi| \leq \ell+1} \| K_i(x) \| \right) \tilde{a}_{T,\ell} + C_1 \left( 1 + \sup_{t \leq T} \sup_{|\xi| \leq \ell+1} |\varphi_i(x)| \right) \tilde{a}_{T,\ell}.
\]

By the triangular inequality and using (2.14), (2.16), we get
\[
|\tilde{A}_0(t, x) - \tilde{A}_0(t, y)| \leq C_{T,\ell}|x - y| r(|x - y|^2).
\]

(2.11) follows in this case. If \( |x| \wedge |y| > \ell + 1 \), the inequality (2.11) holds obviously. If \( |x| > \ell + 1 \geq |y| \) and \( |x - y| \leq \delta_{T,\ell} \), then we can choose \( z \in \mathbb{R}^d \) such that \( z \) is at the intersection of the line determined by \( x, y \) and the sphere centered at the origin of radius \( \ell + 1 \). We have \( |z| = \ell + 1 \), \( |z - y| < |x - y| \) and
\[
|\tilde{A}_0(t, x) - \tilde{A}_0(t, y)| = |\tilde{A}_0(t, z) - \tilde{A}_0(t, y)| \leq C_{T,\ell}|z - y| r(|z - y|^2) \leq C_{T,\ell}|z - y| r(|z - y|^2)
\]
as \( s \to sr(s^2) \) is increasing on \([0, c_0]\). Therefore, the proof of (2.11) is completed. \( \square \)

Let \( Y_i^\ell \) be the solution of
\[
dY_i^\ell = \tilde{A}_0(t, Y_i^\ell), \quad Y_i^\ell = x.
\]

Then by Theorem 2.1, for each \( t > 0 \), \( x \to Y_i^\ell(x) \) is a homeomorphism of \( \mathbb{R}^d \).

**Proof of Theorem 1.** By Lemma 2.2, the solutions \( Y_i(x) \) of (2.6) do not explode at a finite time. We prove now \( (t, x) \to Y_i(x) \) is continuous. For fixed \( w \in \Omega_0 \), \( T \geq 0 \) and arbitrary but fixed \( x_0 \in \mathbb{R}^d \), there exists a constant \( \ell > 1 \) such that \( \sup_{0 \leq s \leq T} |Y_i(x_0)| \leq \ell - 1 \). Then \( Y_i(x_0) \) satisfies the differential equation (2.17). By uniqueness of solutions, \( Y_i(x_0) = Y_i^\ell(x_0) \) for all \( t \in [0, T] \). For any \( 0 < \epsilon < 1 \), there exists \( \delta > 0 \) such that for any \( |x - x_0| < \delta \), one has \( \sup_{0 \leq s \leq T} |Y_i^\ell(x) - Y_i^\ell(x_0)| < \epsilon \), hence,
\[
\sup_{0 \leq s \leq T} |Y_i^\ell(x)| \leq \sup_{0 \leq s \leq T} |Y_i^\ell(x) - Y_i^\ell(x_0)| + \sup_{0 \leq s \leq T} |Y_i^\ell(x_0)| < \epsilon + \ell - 1 < \ell.
\]

Thus, for all \( x \) such that \( |x - x_0| < \delta \) and \( t \in [0, T] \), we have \( Y_i(x) = Y_i^\ell(x) \). It follows that \( (t, x) \to Y_i(x) \) is continuous at \( (t_0, x_0) \in [0, T] \times \mathbb{R}^d \). For the existence of the inverse map of \( x \to Y_i(x) \), it is sufficient to note that the time dependent vector field \( x \to -A_0(T - t, x) \) enjoys the same property as \( x \to A_0(t, x) \).

\( \square \)
Remark 2.4. By Euler approximation, it is seen ([5, 7]) that \( t \to Y^t_\xi(x) \) is adapted to the filtration \( \mathcal{F}_t \) generated by the Brownian motion \( \{w_s; s \leq t\} \), so does \( Y_t(x) \).

3. WONG–ZAKAI APPROXIMATION

The main objective is to prove Theorem 2. First we give the following preparation.

Lemma 3.1. (i) Let \( f: [0, T_0] \to \mathbb{R}_+ \) be an increasing concave function such that \( f(0) = 0 \). Then for \( s \in [0, T_0] \),

\[
\begin{align*}
&f(T_0 \land (Cs)) \leq Cf(s), \quad \text{if } C > 1 \\
&f(Cs) \geq Cf(s), \quad \text{if } 0 \leq C \leq 1.
\end{align*}
\]

(ii) Let \( f \) be a \( C^1 \)-function defined on a neighborhood of 0 such that \( \lim \inf_{s \to 0} f'(s) > 1 \). Then there exists \( \eta > 0 \) such that

\[
f(s) + a \leq f(s + a) \quad \text{for } s \geq 0, \quad a \geq 0, \quad s + a \leq \eta.
\]

Proof. (i) By concavity, for any \( 0 < t_1 < t_2 \),

\[
\frac{f(t_2) - f(0)}{t_2} \leq \frac{f(t_1) - f(0)}{t_1}.
\]

For \( C > 1 \) such that \( Cs \leq T_0 \), taking \( t_2 = Cs \) and \( t_1 = s \) in (3.3) gives \( f(Cs) \leq Cf(s) \); if \( Cs > T_0 \), then \( T_0/C < s \), so that \( f(T_0) = f(CT_0/C) \leq Cf(T_0/C) \leq Cf(s) \). Now for \( 0 \leq C \leq 1 \), taking \( t_1 = Cs \) and \( t_2 = s \) in (3.3) gives the result.

(ii) Since \( \lim \inf_{s \to 0} f'(s) > 1 \), there exists \( \eta > 0 \) small enough such that \( f'(s) > 1 \) for \( s \in [0, \eta] \). Then for \( a > 0, s > 0 \) such that \( s + a \leq \eta \), by mean value theorem, \( f(s + a) - f(s) = f'(\xi) a \geq a \) where \( \xi \in [s, s + a] \); so (ii) holds true.

Remark 3.2. If we extend the function \( f \) on the whole half line \([0, +\infty[\) by setting \( f(s) = f(s \land T_0) \), then \( f \) enjoys (i) and (ii) in Lemma 3.1.

As what was indicated in Introduction, the usual method in [1] or [8] does not work for proving Theorem 2. In the sequel, we will develop the method in [3].

Let \( n \geq 1 \) be an integer. Consider the differential equations

\[
d\tilde{X}^n_t = \left[ \sum_{i=1}^{N} A_i(\tilde{X}^n_t) \tilde{u}_i(t) \right] dt, \quad \tilde{X}^n_0 = x.
\]
Set $\varphi_t^n(x) = \tilde{X}_t^n(x)$, $J_t^n(x) = \partial_x \varphi_t^n(x)$, and $K_t^n(x) = (J_t^n(x))^{-1}$. Define

$$\tilde{A}_0^n(t, x) = K_t^n(x) A_0(\varphi_t^n(x)).$$

(3.4)

For each $a > 1$ and $b > 0$, there exists a family of random variables $\{\xi_n; n \geq 1\}$ such that (see [17] or [1]) for $t \leq 1$ and $x \in \mathbb{R}^d$,

$$|\varphi_t^n(x)| \leq \xi_n(1 + |x|^a), \quad |K_t^n(x)| \leq \xi_n(1 + |x|^b), \quad |J_t^n(x)| \leq \xi_n(1 + |x|^b);$$

moreover,

$$\sup_n \|\xi_n\|_{L^p} < +\infty \quad \text{for any } p > 1. \quad (3.5)$$

Lemma 3.3. Under the conditions in Theorem 1, there exists $\varepsilon_1 > 0$ and a sequence of random variables $\Phi_n$ bounded in all $L^p(\Omega)$ such that

$$|\tilde{A}_0^n(t, x)| \leq \Phi_n \cdot (1 + |x|^{1 - \varepsilon_1}), \quad (t, x) \in [0, 1] \times \mathbb{R}^d \quad (3.7)$$

Proof. Having (3.5) and (3.6) in mind, the same proof as for Lemma 2.2 does work. □

Let $Y_t^n(x)$ be the solution to the differential equations

$$dY_t^n = \tilde{A}_0^n(t, Y_t^n)dt, \quad Y_0^n = x. \quad (3.8)$$

Then $X_t^n = \varphi_t^n(Y_t^n)$ is the solution to the differential equation (1.4).

Lemma 3.4. Let $p > d$ and $M \geq 1$. Then there exists a constant $C > 0$ independent of $n$ such that for $s, t \in [0, 1]$

$$E\left(\sup_{|x| \leq M} |\varphi_t^n(x) - \varphi_s^n(x)|^p\right) \leq C2^{-np},$$

$$E\left(\sup_{|x| \leq M} |\varphi_t^n(x) - \varphi_s^n(x)|^p\right) \leq C|t - s|^p. \quad (3.9)$$

The same results hold for $J_t^n$ and $K_t^n$.

Proof. It was more and less done in [1]. For a detailed proof, we refer to [3; p. 98]. □

Note that from the second estimate in (3.9), we deduce that

$$\sup_n E\left(\sup_{t \leq 1} \sup_{|x| \leq M} |\varphi_t^n(x)|^p\right) < +\infty \quad \text{for any } p > 1.$$
Let $\ell \geq 1$. Introduce the functions $\gamma_\ell \in C^\infty_0(\mathbb{R}^{d \times d}, \mathbb{R}^{d \times d})$ such that $\|\gamma_\ell\|_\infty \leq \ell + 1$ and

$$
\gamma_\ell(u) = \begin{cases} 
    u & \text{if } \|u\| \leq \ell, \\
    0 & \text{if } \|u\| > \ell + 1. 
\end{cases} \quad (3.10)
$$

Similarly we introduce $\rho_\ell \in C^\infty_0(\mathbb{R}^{d \times d}, \mathbb{R}^{d \times d})$. Define

$$
\tilde{A}_0(t, x) = \gamma_\ell(K^\alpha(x))A_0(\rho_\ell(\Phi^\gamma(x))), \quad \tilde{A}_0(t, x) = \gamma_\ell(K^\gamma(x))A_0(\rho_\ell(\Phi^\gamma(x))). \quad (3.11)
$$

Obviously

$$
|\tilde{A}_0(t, x)| \leq C_1(\ell + 2)^2, \quad |\tilde{A}_0(t, x)| \leq C_1(\ell + 2)^2,
$$

for $(t, x) \in [0, 1] \times \mathbb{R}^d$. \quad (3.12)

Let $Y_0^{n, t}$ and $Y_0^t$ be, respectively, the solution to

$$
dY_0^{n, t} = \tilde{A}_0(t, Y_0^{n, t}) dt, \quad Y_0^{n, t} = x \quad (3.13)
$$

and

$$
dY_0^t = \tilde{A}_0(t, Y_0^t) dt, \quad Y_0^t = x. \quad (3.14)
$$

In the sequel, we will denote

$$
m_\ell = C_1(\ell + 2)^2 + \ell \quad \text{and} \quad M_\ell = \frac{2m_\ell + 1}{c_0} \quad (3.15)
$$

where $c_0$ was given in Theorem 1.

**Lemma 3.5.** Let $R > 0$ and $\ell \geq R$. Then for each $t \in [0, 1]$,

$$
\sup_{|x| \leq R} |Y_0^{n, t}(x)| \leq m_\ell, \quad \sup_{|x| \leq R} |Y_0^t(x)| \leq m_\ell. \quad (3.16)
$$

**Proof.** It follows from (3.12) and differential equations (3.13) and (3.14). \qed

In what follows, we consider always $\ell \geq R$. Let $B(M) = \{x \in \mathbb{R}^d; |x| \leq M\}.$

**Lemma 3.6.** Let $f_\ell : [0, +\infty[ \to \mathbb{R}_+$ be defined by

$$
f_\ell(s) = (s \wedge (2m_\ell))r\left(\frac{s \wedge (2m_\ell)}{M_\ell}\right). \quad (3.17)
$$

Then under the condition (ii) in Theorem 1,

$$
|A_0(y_1) - A_0(y_2)| \leq C_2 f_\ell(|y_1 - y_2|), \quad y_1, y_2 \in B(m_\ell). \quad (3.18)
$$
Proof. Note that for the function $r$ given in Theorem 2, there exists a constant $C > 0$ such that $r(s) \leq r(s^2) \leq Cr(s)$ for $s$ small enough. Let $y_1, y_2 \in B(m_i)$. Suppose that $|y_1 - y_2| \leq \epsilon_0$. Up to a change of constant, we have

$$|A_0(y_1) - A_0(y_2)| \leq C_2 |y_1 - y_2| r(|y_1 - y_2|) \leq C_2 |y_1 - y_2| r\left(\frac{|y_1 - y_2|}{M_{t/2}}\right).$$

If $|y_1 - y_2| > \epsilon_0$, then there exists an integer $n_0 > 1$ such that $|y_1 - y_2|/n_0 \leq \epsilon_0$ and $\frac{|y_1 - y_2|}{n_0} > \epsilon_0$ so that $n_0 \leq \frac{|y_1 - y_2|}{\epsilon_0} + 1 \leq M_{t/2}$. Set $z_i = y_1 + \frac{1}{n_0} (y_2 - y_1)$, $i = 0, 1, \ldots, n_0$. Then

$$|A_0(y_1) - A_0(y_2)| \leq \sum_{i=1}^{n_0} |A_0(z_{i-1}) - A_0(z_i)| \leq C_2 \sum_{i=1}^{n_0} |y_1 - y_2| \frac{|y_1 - y_2|}{n_0}. \tag{3.18}$$

which is equal to

$$C_2 |y_1 - y_2| r\left(\frac{|y_1 - y_2|}{n_0}\right) \leq C_2 |y_1 - y_2| r\left(\frac{|y_1 - y_2|}{M_{t/2}}\right),$$

so we get (3.18). \hfill \Box

**Lemma 3.7.** There exists a constant $C_i > 0$ independent of $n$ such that for $y \in \mathbb{R}^d$,

$$|\tilde{A}_0^{t,i}(t, y) - \tilde{A}_0^i(t, y)| \leq C_i (\|K_n^i(y) - K_i(y)\| + f_r(|\varphi_i^*(y) - \varphi_i(y)|)). \tag{3.19}$$

**Proof.** We have $|\rho_i(\varphi_i^*(y)) - \rho_i(\varphi_i(y))| \leq \|\rho_i\|_{\infty} |\varphi_i^*(y) - \varphi_i(y)|$. Since $f_r$ is increasing and by (3.18),

$$|A_0(\rho_i(\varphi_i^*(y))) - A_0(\rho_i(\varphi_i(y)))| \leq C_2 |A_0(\rho_i(\varphi_i^*(y))) - A_0(\rho_i(\varphi_i(y)))| \leq C_2 \|\rho_i\|_{\infty} f_r(|\varphi_i^*(y) - \varphi_i(y)|),$$

this last inequality being deduced from (3.1). Now

$$\tilde{A}_0^{t,i}(t, y) - \tilde{A}_0^i(t, y) = (\gamma_i(K_n^i(y)) - \gamma_i(K_i(y))) \cdot A_0(\rho_i(\varphi_i^*(y))) + \gamma_i(K_i(y)) (A_0(\rho_i(\varphi_i^*(y))) - A_0(\rho_i(\varphi_i(y)))).$$

It follows that

$$|\tilde{A}_0^{t,i}(t, y) - \tilde{A}_0^i(t, y)| \leq C_i (\ell + 2) \|\gamma_i\|_{\infty} \|K_n^i(y) - K_i(y)\|$$

$$+ C_2 (\ell + 1) \|\rho_i\|_{\infty} f_r(|\varphi_i^*(y) - \varphi_i(y)|)$$

which gives (3.19). \hfill \Box
Lemma 3.8. Let
\[
\Lambda_{1,\ell} = \sup_{r \in \mathbb{R}} \sup_{|y| \leq m_{\ell}} \| \hat{\omega}_{r} K_{y}(y) \| \quad \text{and} \quad \Lambda_{2,\ell} = \sup_{r \in \mathbb{R}} \sup_{|y| \leq m_{\ell}} \| J_{y}(y) \|. \tag{3.20}
\]
Then there exists a constant \( C_{\ell} > 0 \) such that
\[
\left| \tilde{A}_{\ell}^{\prime}(t, y_{1}) - \tilde{A}_{\ell}^{\prime}(t, y_{2}) \right| \\
\leq C_{\ell} \left\{ \Lambda_{1,\ell} \cdot |y_{1} - y_{2}| + \Lambda_{2,\ell} \cdot f_{\ell}(|y_{1} - y_{2}|) \right\}, \quad y_{1}, y_{2} \in B(m_{\ell}).
\]

Proof. A direct computation gives the result. \( \square \)

Proposition 3.9. Let \( \eta_{t} = \sup_{|x| \leq R} |v_{t,\ell} \cdot (Y_{r}^{n,\ell}(x) - Y_{r}(x))|^{4p} \), and denote by \( h_{t} = E(\eta_{t}/M_{\rho}^{\ell}) \). Then for any \( p \geq d, t \rightarrow h_{t} \) is Lipschitzian and there exists a constant \( C > 0 \) independent of \( n \) such that
\[
h_{t}^{4} \leq C(h_{t} r(h_{t}) + f_{\ell}(2^{-n/4})), \quad t \leq 1. \tag{3.22}
\]

Proof. Let \( \xi_{t}(x) = |v_{t,\ell} \cdot (Y_{r}^{n,\ell}(x) - Y_{r}(x))|^{2p} \). By (3.16) and (3.12), we know that
\[
\left| \frac{d\xi_{t}(x)}{dt} \right| \leq 8m_{\ell}^{2} \quad \text{for each} \ t \in [0, 1] \text{and}
\]
\[
\left| \frac{d\xi_{t}^{4p}(x)}{dt} \right| \leq 2p2^{2p-1}(x) \left| \frac{d\xi_{t}(x)}{dt} \right| \leq 2p(2m_{\ell})^{4p-2}8m_{\ell}^{2}.
\]

Therefore there exists a constant \( C_{p,\ell} > 0 \) independent of \( n \) and of \( x \) such that for \( |x| \leq R, |\xi_{t}^{4p}(x) - \xi_{s}^{4p}(x)| \leq C_{p,\ell}|t - s| \). Hence,
\[
|\eta_{t} - \eta_{s}| \leq \sup_{|x| \leq R} |\xi_{t}^{4p}(x) - \xi_{s}^{4p}(x)| \leq C_{p,\ell}|t - s|
\]
so that \( t \rightarrow h_{t} \) is a Lipschitzian function. Now we prove the inequality (3.22). We have
\[
\frac{d\xi_{t}(x)}{dt} = 2[v_{t,\ell}(Y_{r}^{n,\ell} - Y_{r})(x, t, Y_{r}^{n,\ell}) - \tilde{A}_{\ell}^{\prime}(t, Y_{r}^{n,\ell})].
\]

Using Lemmas 3.7 and 3.8 and according to (3.16), up to a change of constant \( C_{\ell} > 0 \),
\[
\left| \tilde{A}_{\ell}^{\prime}(t, Y_{r}^{n,\ell}) - \tilde{A}_{\ell}^{\prime}(t, Y_{r}^{n,\ell}) \right| \leq \left| \tilde{A}_{\ell}^{\prime}(t, Y_{r}^{n,\ell}) - \tilde{A}_{\ell}^{\prime}(t, Y_{r}^{n,\ell}) \right| \\
+ \left| \tilde{A}_{\ell}^{\prime}(t, Y_{r}^{n,\ell}) - \tilde{A}_{\ell}^{\prime}(t, Y_{r}^{n,\ell}) \right|
\]
\[ \leq C_\ell \left\{ \sup_{|y| \leq m_s} |K^x_y (y) - K_y(y)| \\
+ f_t \left( \sup_{|y| \leq m_s} |\varphi^y_t (y) - \varphi_t(y)| \right) \\
+ \Lambda_1 \ell |Y^{n, t}_t - Y^t_t| + \Lambda_2 \ell f_x \left( |Y^{n, t}_t - Y^t_t| \right) \right\}. \]

Hence,

\[
\left| \frac{d\xi^{2p}_t}{dt} \right| \leq 4p \xi^{2p-1}_t v \left| Y^{n, t}_t - Y^t_t \right| \\
\times \left\{ \sup_{|y| \leq m_s} |K^x_y (y) - K_y(y)| + f_t \left( \sup_{|y| \leq m_s} |\varphi^y_t (y) - \varphi_t(y)| \right) \\
+ \Lambda_1 \ell v + \Lambda_2 \ell f_x \left( |Y^{n, t}_t - Y^t_t| \right) \right\}
\]

On the subset \( \{ w; v_{t,q}(w) \neq 0 \} \), \( \Lambda_1 \leq q + 1 \) and \( \Lambda_2 \leq q + 1 \). Furthermore, by (3.1), \( v_{t,q} f_x (|Y^{n, t}_t - Y^t_t|) \leq f_t (v_{t,q} |Y^{n, t}_t - Y^t_t|) \). Therefore, it exists a constant \( C_{p,q,t} > 0 \) such that

\[
\left| \frac{d\xi^{2p}_t}{dt} \right| \leq C_{p,q,t} \left\{ \sup_{|y| \leq m_s} |K^x_y (y) - K_y(y)| + f_t \left( \sup_{|y| \leq m_s} |\varphi^y_t (y) - \varphi_t(y)| \right) \\
+ \frac{ \xi^{2p}_t }{ \xi^{q+1}_t } + \frac{ \xi^{2p}_t }{ \xi^{q+1}_t } \left( \frac{ \xi^{2p}_t }{ M^{1p}_t } \right) \right\}.
\]

Hence, for \( 0 \leq s < t \leq 1 \),

\[
\xi^{2p}_t (x) - \xi^{2p}_s (x) \\
\leq C_{p,q,t} (t-s) \left\{ \sup_{|y| \leq m_s} |K^x_y (y) - K_y(y)| + f_t \left( \sup_{|y| \leq m_s} |\varphi^y_t (y) - \varphi_t(y)| \right) \right\} \\
+ C_{p,q,t} \left\{ \int_s^t \eta_u \alpha^{j} du + \int_s^t \eta_u r (\eta_u / M^{1p}_t) du \right\}.
\]

Since \( \sup_{|y| \leq R} \xi^{2p}_y (x) - \sup_{|y| \leq R} \xi^{2p}_y (x) \leq \sup_{|y| \leq R} (\xi^{2p}_y (x) - \xi^{2p}_y (x)) \), the above inequality leads to

\[
\eta_t - \eta_s \leq C_{p,q,t} (t-s) \left\{ \sup_{|y| \leq m_s} |K^x_y (y) - K_y(y)| + f_t \left( \sup_{|y| \leq m_s} |\varphi^y_t (y) - \varphi_t(y)| \right) \right\} \\
+ C_{p,q,t} \left\{ \int_s^t \eta_u \alpha^{j} du + \int_s^t \eta_u r (\eta_u / M^{1p}_t) du \right\}.
\]
Taking the expectation and up to a change of constant, we have
\[
ht - hs \leq C_{p,q,\ell}(t - s) \left\{ E\left( \sup_{|y| \leq m_t} |K^+_t(y) - K_t(y)| \right) \right. \\
+ f_r \left( E\left( \sup_{|y| \leq m_t} |\varphi^+_t(y) - \varphi_t(y)| \right) \right) \right. \\
+ \left. C_{p,q,\ell} \int_s^t h_u du + \int_s^t h_u r(h_u) du \right\}.
\]
Taking the right derivative and using Lemma 3.4, we get
\[
h'_t \leq C_{p,q,\ell}\left\{ ht + h_t r(h_t) + f_r(2^{-n/4}) + 2^{-n/4} \right\}
\]
which implies (3.22) with a different constant. □

Proposition 3.10. For \( p, \ell, q \) given, there exists a sequence \( \alpha_n \geq 0 \) such that \( \sum \alpha_n < +\infty \) and
\[
E\left( \sup_{|x| \leq R} \left| Y^{n,\ell}_t(x) - Y^{\ell}_t(x) \right|^{4p} \right) \leq M_t \alpha_n, \quad t \leq 1. \tag{3.23}
\]
Proof. Note that for \( p \) and \( n \) big enough, \( h_t \) and \( f_r(2^{-n/4}) \) are sufficiently small. By (3.2), we have
\[
h_t r(h_t) + a_n \leq (h_t + a_n) r(h_t + a_n)
\]
where \( a_n = f_r(2^{-n/4}) \). Let \( \tilde{h}_t = h_t + a_n \). Then, by (3.22), we have
\[
\tilde{h}'_t \leq C_{p,q,\ell} \tilde{h}_t r(\tilde{h}_t), \quad \tilde{h}_0 = a_n. \tag{3.24}
\]
To solve (3.24), we introduce
\[
e_0(s) = s, \quad e_1(s) = e^s, \quad e_k = \exp(e_{k-1}(s)) = e_{k-1}(e^s).
\]
Recall that \( r(s) = \log \frac{s}{2} \log \frac{1}{2} \cdots \log \frac{1}{2} \). By (3.24), we have
\[
\int_0^t \frac{\tilde{h}'_s ds}{\tilde{h}_s \log \frac{1}{\tilde{h}_s} \cdots \log \frac{1}{\tilde{h}_s}} \leq Ct.
\]
Letting \( u = \log \frac{1}{h} \), in the above inequality gives
\[
- \int_{\log(1/h)}^{\log(1/a_n)} \frac{du}{u \log u \cdots \log_{k-1}(u)} \leq Ct,
\]
from which we get
\[ \log \left( \frac{\log(1/h_t)}{\log(1/a_n)} \right) \geq -C t, \]
or
\[ \log_{k-1} \left( \frac{1}{h_t} \right) \geq \left( \log_{k-1} \left( \frac{1}{a_n} \right) \right)^{e^{-C t}}. \]

We deduce that
\[ \tilde{h}_t \leq \exp \left[ -e_{k-2} \left( \left( \log_{k-1} \left( \frac{1}{a_n} \right) \right)^{e^{-C t}} \right) \right]. \tag{3.25} \]

By definition of \( a_n \), there exists \( 2^{1/4} \geq \zeta > 1 \) (which is dependent of \( \ell \)) such that \( a_n \leq \zeta^{-n} \) or \( 1/a_n \geq \zeta^n \). Let \( 1 > \gamma_1 = \log \zeta > 0 \). Then for \( k \geq 2 \), there exists \( n_{k,t} \geq 1 \) such that
\[ \log_{k-1}(\zeta^n) \geq \gamma_1 \cdot \log_{k-2}(n) \geq 1, \quad \text{for } n \geq n_{k,t}. \]

Hence, by (3.25), we get
\[ h_t \leq \exp \left( -e_{k-2} \left( (\gamma_1 \log_{k-2}(n))^{e^{-C t}} \right) \right) \tag{3.26} \]
where \( \beta = e^{-C} \). Let \( \varepsilon_n \) be the right hand side of (3.26). Then for any integer \( N_0 \geq 1 \),
\[ \varepsilon_n \leq \frac{1}{\mu^{N_0}} \quad \text{for } n \text{ big enough.} \tag{3.27} \]

In fact, it is obvious that for \( n \) big enough,
\[ (\gamma_1 \log_{k-2}(n))^{e^{-C t}} \geq N_0 \log \left( \log_{k-2}(n) \right) \geq \log_{k-2}(N_0 \log n). \]

Now we complete the proof of (3.23) with (3.26) and (3.27). \( \square \)

**Proposition 3.11.** Set \( \psi_{p,q}^{n,\ell} = \rho_\ell \circ \varphi_{p}^{n,\ell}, \psi_{t}^{n} = \rho_\ell \circ \varphi_{t} \). Then for \( p, q, \ell \) given, there exists a constant \( C > 0 \) independent of \( n \) such that
\[ E \left\{ \sup_{|x| \leq R} \left| \psi_{p,q}^{n,\ell}(Y_{n,t}^{n,\ell}(x)) - \psi_{t}^{n,\ell}(Y_{t}^{n,\ell}(x)) \right|^q \right\} \leq C \varepsilon_n, \quad t \in [0, 1]. \tag{3.28} \]

**Proof.** We have \( |\psi_{p,q}^{n,\ell}(Y_{n,t}^{n,\ell}(x)) - \psi_{t}^{n,\ell}(Y_{t}^{n,\ell}(x))| \leq \| \rho_\ell \|_\infty \left( \sup_{|y| \leq \mu} |\varphi_p^{n,\ell}(y) - \varphi_t(y)| \right) \). By Lemmas 3.4 and 3.5
\[ E \left\{ \sup_{|x| \leq R} \left| \psi_{p,q}^{n,\ell}(Y_{n,t}^{n,\ell}(x)) - \psi_{t}^{n,\ell}(Y_{t}^{n,\ell}(x)) \right|^q \right\} \leq C_{\ell,p} \cdot 2^{-np}. \tag{i} \]
On the other hand,
\[ v_{\ell,q} |\psi^t_s(Y^n_\ell(x)) - \psi^t_s(Y_\ell(x))| \leq v_{\ell,q} \| \psi^t_s \|_\infty \left( \sup_{|y| \leq m_y} |Y^n_\ell(x) - Y_\ell(x)| \right) \]
which is less than \( C_{\ell,q} v_{\ell,q} |Y^n_\ell(x) - Y_\ell(x)| \), as \( \sup_{|y| \leq m_y} |J_s(y)| \leq q + 1 \) for \( v_{\ell,q} \neq 0 \). Using (3.23),
\[ E \left( \sup_{|x| \leq R} |v_{\ell,q} (\psi^t_s(Y^n_\ell(x)) - \psi^t_s(Y_\ell(x)))|^{4p} \right) \leq C_{p,q,\ell} |t - s|^p. \]  
(ii)

Now note that
\[ v_{\ell,q} (\psi^t_s(Y^n_\ell(x)) - \psi^t_s(Y_\ell(x))) = v_{\ell,q} (\psi^t_s(Y^n_\ell(x)) - \psi^t_s(Y_\ell(x))) + v_{\ell,q} (\psi^t_s(Y_n^\ell(x)) - \psi^t_s(Y_\ell(x))). \]
Combining (i) and (ii), we get (3.28).

\[ \square \]

**Proposition 3.12.** For each \( p \geq d \) and \( q, \ell \) given, there exists a constant \( C_{p,q,\ell} > 0 \) independent of \( n \) such that
\[ E \left( \sup_{|x| \leq R} |v_{\ell,q} (\psi^t_s(Y^n_\ell(x)) - \psi^t_s(Y_\ell(x)))|^{4p} \right) \leq C_{p,q,\ell} |t - s|^p. \]  
(3.29)

**Proof.** We have
\[ v_{\ell,q} (\psi^t_s(Y^n_\ell(x)) - \psi^t_s(Y_\ell(x))) = v_{\ell,q} (\psi^t_s(Y_n^\ell(x)) - \psi^t_s(Y_\ell(x))) + v_{\ell,q} (\psi^t_s(Y_n^\ell(x)) - \psi^t_s(Y_n^\ell(x))). \]
For the first term of the right-hand side,
\[ |\psi^t_s(Y_n^\ell(x)) - \psi^t_s(Y_\ell(x))| \leq \| \psi^t_s \|_\infty \left( \sup_{|x| \leq m_y} |Y_n^\ell(x)| \right) \]  
\[ |Y_n^\ell(x)| \leq C_\ell |t - s| \]  
for some constant dependent of \( \ell \). Therefore,
\[ E \left( \sup_{|x| \leq R} |\psi^t_s(Y_n^\ell(x)) - \psi^t_s(Y_\ell(x))|^{4p} \right) \leq C_{p,\ell} |t - s|^{4p}, \]  
(i) as (see Lemma 3.4) \( \sup_{|x| \leq R} E \left( \sup_{|x| \leq m_y} |J_s(x)|^{4p} \right) < +\infty \). On the other hand,
\[ |\psi^t_s(Y_n^\ell(x)) - \psi^t_s(Y_\ell(x))| \leq \| \psi^t_s \|_\infty \sup_{|x| \leq m_y} |\varphi^\ell(x) - \varphi^\ell(x)|. \]
Again, by Lemma 3.4, we get
\[ E \left( \sup_{|x| \leq R} |\psi^t_s(Y_n^\ell(x)) - \psi^t_s(Y_\ell(x))|^{4p} \right) \leq C_{\ell,q} |t - s|^p. \]  
(ii)
Combining (i) and (ii) gives (3.29).
\[ \square \]
Now fix a $p_0 > d$ big enough and set $b_n = s_n^{1/(4b)}$. Let $s_0 = 0$, $s_1 = b_1$, $s_n = \sum_{j=1}^{n} b_j$, and $s_\infty := \sum_{j=1}^{\infty} b_j$ is finite according to (3.27). Set $X_n^{\ell,q}(t, x) = \psi_{\ell,q}(Y_n^{\ell,q}(x))$. We define

$$\tilde{X}^{\ell,q}(s, t, x) = X_n^{\ell,q}(t, x) + (s - s_n) \frac{X_n^{\ell,q}(t, x) - X_n^{\ell,q}(s, x)}{s_{n+1} - s_n}$$

for $s \in [s_n, s_{n+1}]$.

Then $(s, t) \to \tilde{X}^{\ell,q}(s, t, \cdot)$ is random field defined on a compact subset $[0, s_\infty] \times [0, 1]$ taking values in the Banach space $C(B(R), \mathbb{R}^d)$. By Propositions 3.11 and 3.12, we have

$$E\left( \sup_{s \leq R} \left| \tilde{X}^{\ell,q}(s_1, t_1, x) - \tilde{X}^{\ell,q}(s_2, t_2, x) \right|^{4b} \right) \leq C_{\ell,q}(|s_1 - s_2|^{4b} + |t_1 - t_2|^{4b})$$

(3.30)

**Proposition 3.13.** Let $X^{\ell,q}(t, x) = \psi_{\ell,q}(Y^{\ell,q}(x))$. There exists a subset $\Omega^{\ell,q} \subset \Omega$ such that $P(\Omega^{\ell,q}) = 1$ and for $w \in \Omega^{\ell,q}$, $X_n^{\ell,q}(t, x)$ converges to $X^{\ell,q}(t, x)$ uniformly with respect to $(t, x) \in [0, 1] \times B(R)$.

**Proof.** By Kolmogorov modification theorem, $\tilde{X}^{\ell,q}(s, t, \cdot)$ has a continuous version, that we use the same notation. We have $X_n^{\ell,q}(t, x) = \tilde{X}^{\ell,q}(s_n, t, x)$, which converges uniformly relative to $(t, x) \in [0, 1] \times B(R)$ as $n \to +\infty$. □

To prove Theorem 2, we need the last preparation.

**Proposition 3.14.** Let $Y_i$ be defined in (2.6). Then for any $p > 1$,

(i) $E(\sup_{t \geq 1} \sup_{|x| \leq R} |Y_i(x)|^p) < +\infty$,

(ii) $E(\sup_{t \geq 1} \sup_{|x| \leq R} |\varphi_i(Y_i(x))|^p) < +\infty$, and

(iii) $E(\sup_{t \geq 1} \sup_{|x| \leq R} |K_i(Y_i(x))|^p) < +\infty$.

**Proof.** Let $x \in B(R)$. Applying (2.9), $|Y_i(x)| \leq R + \Phi_1 \int_0^t (1 + |Y_i(x)|^{1-\varepsilon_i}) ds$ for each $t \leq 1$. Let $y_i = \sup_{t \geq 1} \sup_{|x| \leq R} |Y_i(x)|$. By above inequality, we have

$$y_i \leq R + \Phi_1 \int_0^t (1 + y_i^{1-\varepsilon_i}) ds.$$

It follows that for some constant $C_\rho > 0$,

$$E(y_i^\rho) \leq C_\rho \left( 1 + \exp \left[ C_\rho \left( \Phi_1^{\rho/(\varepsilon_i)} \right)^{\varepsilon_i} / (1 - \varepsilon_i) \right] \right).$$

(3.31)

The item (i) follows. The items (ii) and (iii) deduce from (i) and (2.8). □
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Now by (3.5), (3.7), and (3.31), we see that the following hold

\[
\sup_n \mathbb{E}\left( \sup_{|t| \leq R} |Y^n_t(x)|^p \right) < +\infty, \quad \sup_n \mathbb{E}\left( \sup_{|t| \leq R} |\varphi^n_t(Y^n_t(x))|^p \right) < +\infty. 
\]

(3.32)

\[
\sup_n \mathbb{E}\left( \sup_{|t| \leq R} |K^n_t(Y^n_t(x))|^p \right) < +\infty. 
\]

(3.33)

**Proof of Theorem 2.** For \( \ell \geq R \geq 1 \) and \( q \geq 1 \) given, we set

\[
\Gamma_\ell = \left\{ w; \sup_{|t| \leq R} |\varphi_t(Y_t(x))| \leq \ell, \sup_{|t| \leq R} |K_t(Y_t(x))| \leq \ell \right\};
\]

similarly we define \( \Gamma_{n, \ell} \). Let \( \Delta_{q, \ell} = \left\{ w; \Lambda_1(w) + \Lambda_2(w) \leq q \right\}. \) By Proposition 3.14 and (3.32) and (3.33), there exists a constant \( C > 0 \) independent of \( n \) such that

\[
P(\Gamma_\ell) \leq C/\ell^2, \quad P(\Gamma_{n, \ell}) \leq C/\ell^2
\]

(3.34)

and for \( \ell \geq R \) given, \( \lim_{p \to +\infty} P(\Delta_{q, \ell}) = 0 \). Now on \( \Gamma_\ell \cap \Delta_{q, \ell} \), for \( |x| \leq R \) and \( t \leq 1 \), \( Y^n_t(x) = Y_t(x) \) so that \( X_t^q(t, x) = X_t(x, w) \); on the subset \( \Gamma_{n, \ell} \cap \Delta_{q, \ell} \), for \( |x| \leq R \) and \( t \leq 1 \), \( Y^n_t(x) = Y^n_t(x) \) so that \( X_t^q(t, x) = X_t^q(t, x) \). We have

\[
\mathbb{E}\left( \sup_{|t| \leq R} |X^n_t(x) - X_t(x)|^p \right)
\]

\[
\leq \mathbb{E}\left( \sup_{|t| \leq R} |X^n_t(x) - X_t(x)|^p 1_{\Gamma_\ell \cap \Delta_{q, \ell}} \right)
\]

\[
+ \mathbb{E}\left( \sup_{|t| \leq R} |X^n_t(x) - X_t(x)|^p 1_{\Gamma_{n, \ell} \cap \Delta_{q, \ell}} \right)
\]

\[
+ \mathbb{E}\left( \sup_{|t| \leq R} |X^n_t(x) - X_t(x)|^p 1_{\Delta_{q, \ell}} \right).
\]

According to the above discussions, the first term on the right side is equal to

\[
\mathbb{E}\left( \sup_{|t| \leq R} |X^n_t^q(t, x) - X^q(t, x)|^p 1_{\Gamma_\ell \cap \Delta_{q, \ell}} \right) \to 0 \quad \text{as} \ \ n \to +\infty
\]

by Proposition 3.13 and Lebesgue dominated convergence theorem. Using Proposition 3.14 with \( 2p \) and (3.34), the Cauchy–Schwartz inequality implies that the second term on the right side is dominated by \( C/\ell \) where \( C \) is independent of \( n \); while the last term is dominated by \( C/\sqrt{P(\Delta_{q, \ell})} \). Now letting \( q \to +\infty \) and \( \ell \to +\infty \) completes the proof of Theorem 2. \( \Box \)
4. REGULARIZING THE DRIFT

Let $A_0$ be the vector field given in Theorem 2. We will regularize $A_0$. Let $\chi \in C^\infty_0(\mathbb{R}^d)$ such that $0 \leq \chi \leq 1$, $\text{supp} \chi \subset B(1)$ and $\int_{\mathbb{R}^d} \chi(x)dx = 1$. For $n \geq 1$, define $\chi_n(x) = 2^{dn} \chi(2^n x)$. Then $\text{supp} \chi_n \subset B(2^{-n})$ and $\int_{\mathbb{R}^d} \chi_n(x)dx = 1$. Set

$$A^n_0 = A_0 \ast \chi_n \quad \text{and} \quad A^n_0 = \beta_n A^n_0$$

(4.1)

where $\beta_n$ is defined in (2.10) and $\ast$ is the convolution product.

Proposition 4.1.

(i) There is a constant $C > 0$ independent of $n$ such that

$$|A^n_0(x)| \leq C(1 + |x|^{1-\epsilon_0}).$$

(4.2)

(ii) There exists $\zeta > 1$ such that for any $M \geq 1$,

$$\sup_{|x| \leq M} |A^n_0(x) - A_0(x)| \leq \zeta^{-n} \quad \text{for } n \text{ big enough.}$$

(4.3)

Proof. (i) By sub linear growth of $A_0$, we have

$$|A_0(x - y)| \leq C_1(1 + |x - y|^{1-\epsilon_0}) \leq C_1(1 + |x|^{1-\epsilon_0} + |y|^{1-\epsilon_0}).$$

Then

$$|A^n_0(x)| \leq C_1 \int_{\mathbb{R}^d} (1 + |x|^{1-\epsilon_0} + |y|^{1-\epsilon_0})\chi_n(y)dy \leq C_1(2 + |x|^{1-\epsilon_0}).$$

We get (4.2), up to a change of constant. For (ii), we have for each $x \in \mathbb{R}^d$,

$$|A^n_0(x) - \beta_n(x)A_0(x)| \leq \int_{B(2^{-n})} |A_0(x - y) - A_0(x)|\chi_n(y)dy$$

$$\leq C_2 \int_{B(2^{-n})} |y||y|^2\chi_n(y)dy.$$

Since $s \to sr(s^2)$ is increasing, the last term in the above estimate is majorized by $C_22^{-n}r(2^{-2n}) \leq C_2\zeta^{-n}$ for some $1 < \zeta < 2$, where $r$ is defined in Theorem 2. Now for $M \geq 1$ given, when $n \geq M$, $\sup_{|x| \leq M} |\beta_n A_0 - A_0| = 0$. So we get (ii).

Theorem 4.2. Let $\{X_n(t); t \geq 0\}$ be the solution to the SDE

$$dX_n(t) = \sum_{i=1}^N A_i(X_n(t)) \circ dw^i_t + A^n_0(X_n(t))dt, \quad X_n(0) = x.$$
Flow of Homeomorphisms

Then for any $p > 1$ and $R \geq 1$,

$$\lim_{n \to +\infty} E \left( \sup_{t \leq 1} \sup_{|x| \leq R} |X_n(t) - X(t)|^p \right) = 0. \quad (4.5)$$

**Proof.** The proof is similar to that of Theorem 2; we will give a sketch. Let $\varphi_t(x)$ be the flow associated to $d\tilde{X}_t = \sum_{i=1}^{N} A_i(\tilde{X}_t) \circ dw_i$. As in (2.5), set $J_t(x) = (\tilde{\varepsilon}_x \varphi_t(x))$ and $K_t(x) = J_t(x)^{-1}$. Consider $\tilde{A}_0^\ell(t, x) = K_t(x)A_0^\ell(\varphi_t(x))$ and $d\tilde{Y}_t^n = \tilde{A}_0^\ell(t, \tilde{Y}_t^n)dt, \quad \tilde{Y}_0^n = x$.

Then $X_n(t, x) = \varphi_t(\tilde{Y}_t^n(x))$. Using (4.2), as for the proof of Lemma 2.2, for any $T > 0$, there exists $\Phi_T \in \bigcap_{p>1} L^p$ such that

$$|\tilde{A}_0^\ell(t, x)| \leq \Phi_T(1 + |x|^{1-\varepsilon}), \quad (t, x) \in [0, T] \times \mathbb{R}^d. \quad (4.6)$$

Similarly to (3.11), we define

$$\tilde{A}_0^{\ell,n}(x) = \gamma_t(K_t(x))A_0^\ell(\rho_t(\varphi_t(x))), \quad \tilde{A}_0^\ell(x) = \gamma_t(K_t(x))A_0(\rho_t(\varphi_t(x))).$$

Again by (4.2), we have

$$|\tilde{A}_0^{\ell,n}(t, x)| \leq C(\ell + 2)^2 \quad (4.7)$$

where $C$ is independent of $\ell$ and $n$. Let $Y_t^{n,\ell}, \ Y_t^{\ell}$ be respectively solution to

$$dY_t^{n,\ell} = \tilde{A}_0^{\ell,n}(t, Y_t^{n,\ell})dt, \quad Y_0^{n,\ell} = x,$n

$$dY_t^{\ell} = \tilde{A}_0^\ell(t, Y_t^{\ell})dt, \quad Y_0^{\ell} = x.$$n

Then the estimate (3.16) holds for such $Y_t^{n,\ell}$. Now by (4.3), there exists a constant $C_\ell$ (independent of $n$) such that for all $(t, x),

$$|\tilde{A}_0^{\ell,n}(t, x) - \tilde{A}_0(t, x)| \leq C_\ell \varepsilon^{-n}. \quad (4.8)$$

We have

$$\left| \frac{dY_t^{n,\ell}}{dt} - \frac{dY_t^{\ell}}{dt} \right| = \left| \tilde{A}_0^{\ell,n}(t, Y_t^{n,\ell}) - \tilde{A}_0(t, Y_t^{\ell}) \right| \leq \left| \tilde{A}_0^{\ell,n}(t, Y_t^{n,\ell}) - \tilde{A}_0^\ell(t, Y_t^{n,\ell}) \right| + \left| \tilde{A}_0(t, Y_t^{n,\ell}) - \tilde{A}_0(t, Y_t^{\ell}) \right|.$$n

This last quantity is dominated by, according to (4.8) and to Lemma 3.8:

$$C_\ell \varepsilon^{-n} + A_{1,\ell}|Y_t^{n,\ell} - Y_t^{\ell}| + A_{2,\ell}\varepsilon f_\ell(|Y_t^{n,\ell} - Y_t^{\ell}|).$$

From which we see that Proposition 3.9 subsists. Now following the remainder of Section 3, we can complete the proof. $\Box$
5. STOCHASTIC TRANSPORT EQUATIONS

When the coefficients are sufficiently regular, the solutions to some class of stochastic partial differential equations were constructed with the help of stochastic flows of diffeomorphisms associated to SDE (see [10]). The purpose of this section is to apply our preceding results to a study of transport equations.

For the moment, suppose that \( A_0 \in C_0^{2+\delta} \). Let \( X_t(x, w) \) be the solution to the Stratanovich SDE

\[
\frac{dX_t}{dt} = \sum_{i=1}^{N} A_i(X_t) \circ dw_t^i + A_0(X_t)dt, \quad X_0 = x.
\]

By [10], \( x \to X_t(x, w) \) is a \( C_0^{2+\delta} \)-diffeomorphisms, where \( 0 < \delta_0 < \delta \). Moreover (see [1; pp. 103–106])

\[
dX_t^{-1}(x) = -\left[ \left( \frac{\partial}{\partial x} X_t \right) X_t^{-1} \right]^{-1} \left( \sum_{i=1}^{N} A_i(x) \circ dw_t^i + A_0(x)dt \right). \tag{5.1}
\]

For \( \theta_0 \in C^{2+\delta}(\mathbb{R}^d) \), we set

\[
\theta(t, x) = \theta_0(X_t^{-1}(x)). \tag{5.2}
\]

Then almost surely, for each \( t \geq 0 \), \( x \to \theta(t, x) \) is in \( C^{2+\delta_0} \). We have

\[
d\theta(t, x) = -\sum_{i=1}^{N} \left( \nabla \theta(t, x) \cdot A_i(x) \right) \circ dw_t^i - \left( \nabla \theta(t, x) \cdot A_0(x) \right)dt \tag{5.3}
\]

where \( d\theta(t, x) \) denotes the stochastic differential with respect to \( t \) and \( \nabla \theta(t, x) \) the gradient relative to \( x \). Passing to Itô stochastic integrals, (5.3) leads to

\[
d\theta(t, x) + \sum_{i=1}^{N} \left( \nabla \theta(t, x) \cdot A_i(x) \right) \cdot dw_t^i + \left( \nabla \theta(t, x) \cdot A_0(x) \right)dt
\]

\[
- \frac{1}{2} \sum_{i=1}^{N} \left( \mathcal{L} A_i \theta \right)(t, x)dt = 0 \tag{5.4}
\]

where \( \mathcal{L} A_i \) denotes the Lie derivative with respect to \( A_i \).

Now let’s come back to the situation in Theorem 2. Actually by Theorem 1, almost surely for each \( t > 0 \), \( x \to X_t(x, w) \) is a homeomorphism of \( \mathbb{R}^d \). So it is natural to ask whether \( \theta(t, x) = \theta_0(X_t^{-1}(x)) \) solves (5.3) in a weaker sense. To this end, we will suppose that \( A_0 \) is divergence free in distributional sense, that is,

\[
\int_{\mathbb{R}^d} \phi(\nabla \psi, A_0)dx = -\int_{\mathbb{R}^d} \psi(\nabla \phi, A_0)dx, \quad \psi, \phi \in C_0^\infty(\mathbb{R}^d). \tag{5.5}
\]
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Then if $A_0$ is divergence free, the divergence $\text{div}(\phi A_0)$ of $\phi A_0$ has the expression $\text{div}(\phi A_0) = -[\nabla \phi, A_0]$.

**Theorem 5.1.** Assume the same conditions as in Theorem 2 and $A_0$ is divergence free. Let $\theta_0$ be a continuous function on $\mathbb{R}^d$, having polynomial growth:

$$|\theta_0(x)| \leq C(1 + |x|^{q_0}), \quad x \in \mathbb{R}^d. \tag{5.6}$$

Then $\theta(t, x) = \theta_0(X_t^{-1}(x))$ solves the stochastic transport equation (5.3) in distributional sense, that is, for any $\phi \in C_0^\infty(\mathbb{R}^d)$,

$$(\theta_0, \phi)_{L^2} = (\theta_0, \phi)_{L^2} - \sum_{i=1}^N \int_0^T (\theta_0, \text{div}(\phi A_i))_{L^2} \circ d\tilde{w}_t^i - \int_0^T (\theta_0, \text{div}(\phi A_0))_{L^2} ds \tag{5.7}$$

where $\theta_0 := \theta(t, \cdot)$ and $(\cdot, \cdot)_{L^2}$ denotes the inner product in $L^2(\mathbb{R}^d, dx)$.

**Proof.** Fix $T \in [0, 1]$ and consider $\tilde{w}_t = w_t - w_{t-}$, for $0 \leq t \leq T$. Let $\tilde{X}_t(x, \tilde{w}_t)$ be the solution to the following SDE:

$$d\tilde{X}_t = \sum_{i=1}^N A_i(\tilde{X}_t) \circ d(\tilde{w}_t) - A_0(\tilde{X}_t) dt, \quad \tilde{X}_0 = x \tag{5.8}$$

Then applying Theorem 2 to (1.3) and (5.8), respectively, we get the relations

$$X_{T-t}(x, w) = \tilde{X}_t(X_T(x, w), \tilde{w}_T),$$
$$\tilde{X}_{T-t}(x, \tilde{w}_T) = X_t(\tilde{X}_T(x, \tilde{w}_T), w), \quad 0 \leq t \leq T.$$ 

Taking $t = T$ in the above relations gives that $X_T^{-1}(\cdot, w) = \tilde{X}_T(\cdot, \tilde{w}_T)$. So some properties concerning $X_T^{-1}(\cdot, w)$ can be deduced from SDE (5.8). Now let $\{A_0^n; n \geq 1\}$ be a sequence of smooth vector fields defined in (4.1). Replace $A_0$ by $A_0^n$ in (5.8) and let $\{\tilde{X}_t^n; 0 \leq t \leq T\}$ be the associated solution. By Theorem 4.2, for any $R \geq 1$ and $p > 1$,

$$\lim_{n \to +\infty} \mathbb{E} \left( \sup_{t \leq 1} \sup_{|x| \leq R} |\tilde{X}_t^n(x) - \tilde{X}_t(x)|^p \right) = 0.$$ 

Therefore for each $t \in [0, 1]$,

$$\lim_{n \to +\infty} \mathbb{E} \left( \sup_{|x| \leq R} |(X_t^n)^{-1}(x) - X_t^{-1}(x)|^p \right) = 0. \tag{5.9}$$
Step 1. Suppose that $\theta_0 \in C^\infty_0(\mathbf{R}^d)$. Then $\theta_n(t, x) = \theta_0((X_n^t)^{-1}(x))$ solves (5.4) with $A_0$ replaced by $A_n^\phi$. Then for $\phi \in C^\infty_0(\mathbf{R}^d)$, we have

$$
(\theta_n(t), \phi)_{L^2} = (\theta_0, \phi) - \sum_{i=1}^N \int_0^t (\theta_n(s), \text{div}(\phi A_i^\phi))_{L^2} dw_i' \\
- \int_0^t (\theta_n(s), \text{div}(\phi A_n^\phi))_{L^2} ds \\
+ \frac{1}{2} \sum_{i=1}^N \int_0^t (\theta_n(s), \text{div}(A_i \text{div}(\phi A_i)))_{L^2} ds.
$$

As $\sup_{|t| \leq R} |\theta_n(t, x) - \theta(t, x)| \leq (||\theta_0||_\infty) \sup_{|t| \leq R} |(X_n^t)^{-1}(x) - X_r^{-1}(x)|$ and by (5.9), we have

$$
\lim_{n \to +\infty} E\left( \sup_{|t| \leq R} |\theta_n(t, x) - \theta(t, x)|^p \right) = 0.
$$

Now take $R > 0$ big enough such that supp $\phi \subset B(R)$. Then as $n \to +\infty$,

$$
E\left( |(\theta_n(t), \phi)_{L^2} - (\theta(t), \phi)_{L^2}|^2 \right) \\
\leq \left( \int_{\mathbf{R}^d} |\phi(x)|^2 dx \right) E\left( \int_{B(R)} |\theta_n(t, x) - \theta(t, x)|^2 dx \right) \to 0.
$$

Let $\phi_i = \text{div}(\phi A_i)$ for $i = 1, \ldots, N$. Replacing $\phi$ by $\phi_i$ in (5.12), for each $s \in [0, 1]$,

$$
\lim_{n \to +\infty} E((\theta_n(s), \phi_i)_{L^2} - (\theta(s), \phi_i)_{L^2})^2 = 0.
$$

On the other hand, $|(\theta_n(s), \phi_i)_{L^2}| \leq ||\theta_0||_\infty \left( \int_{\mathbf{R}^d} |\phi_i(x)| dx \right)$. By Lebesgue dominated convergence theorem,

$$
\lim_{n \to +\infty} \left[ \int_0^1 \sum_{i=1}^N ((\theta_n(s), \phi_i)_{L^2} - (\theta(s), \phi_i)_{L^2})^2 \right] = 0.
$$

Therefore, $\sum_{i=1}^N \int_0^1 (\theta_n(s), \text{div}(\phi A_i))_{L^2} dw_i'$ converges to $\sum_{i=1}^N \int_0^1 (\theta(s), \text{div}(\phi A_i))_{L^2} dw_i'$ in $L^2(\Omega)$, but uniformly relative to $t \in [0, 1]$. In the same way, the last term on the right side in (5.10) tends to $\frac{1}{2} \sum_{i=1}^N \int_0^1 (\theta(s), \text{div}(A_i \text{div}(\phi A_i)))_{L^2} ds$ in $L^2(\Omega)$. Now since $A_0$ is divergence free so that $\overline{A_0}$ defined in (4.1) is divergence free in classical sense, then

$$
\text{div}(\phi A_n^\phi) = \text{div}(\phi \beta_n, \overline{A_0}) = -\nabla(\phi \beta_n, \overline{A_0}).
$$
which tends to $-\{\nabla \phi, A_0\} = \text{div}(\phi A_0)$. It follows that the term $J_0^t(\theta_s(x), \text{div}(\phi A_0))_{L^2} ds$ converges to $J_0^t(\theta(x), \text{div}(\phi A_0))_{L^2} ds$ in $L^2(\Omega)$. Now taking the limit $n \to +\infty$ in (5.10), we get

$$(\theta(t), \phi)_{L^2} = (\theta_0, \phi)_{L^2} - \sum_{i=1}^N \int_0^t (\theta(s), \text{div}(\phi A_i))_{L^2} du_i$$

$$- \int_0^t (\theta(s), \text{div}(\phi A_0))_{L^2} ds$$

$$+ \frac{1}{2} \sum_{i=1}^N \int_0^t (\theta(s), \text{div}(A_i \text{div}(\phi A_i)))_{L^2} ds.$$  (5.13)

**Step 2.** Suppose that $\theta_0 \in C(\mathbb{R}^d)$ satisfying (5.6). Define $\theta_0^p = \beta_n(\theta_s \ast \chi_n)$. Then there exists a constant $C > 0$ independent of $n$ such that

$$|\theta_0^p(x)| \leq C(1 + |x|^p), \quad x \in \mathbb{R}^d.$$  (5.14)

Use again the notation $\theta_n(t, x)$ to denote $\theta_n(t, x) = \theta_0^p(X_t^{-1}(x))$ where $X_t(x)$ is now the solution to the SDE (1.3). Then $\theta_n$ satisfies (5.13). Now using the SDE (5.8) and the moment estimate (see Proposition 1.3 in [4]), we have for any $T \in [0, 1]$,

$$\mathbb{E}\left( \sup_{\theta \in \mathbb{R}^d \leq T} |\tilde{X}_t(x)|^p \right) \leq C_p(1 + |x|^p).$$

in particular, for each $t \in [0, 1]$,

$$\mathbb{E}(|X_t^{-1}(x)|^p) \leq C_p(1 + |x|^p).$$

By (5.14) and (5.6), it holds that

$$\sup_{r \leq T} \mathbb{E}(|\theta_n(t, x)|^p) + \sup_{r \leq T} \mathbb{E}(|\theta(t, x)|^p) \leq C_p(1 + |x|^p).$$  (5.15)

Therefore for any $p > 2$, $\phi \in C_0^\infty(\mathbb{R}^d)$ with $\text{supp} \phi \subset B(R)$,

$$\int_0^1 \mathbb{E}(|\theta(t), \phi)_{L^2}|^p) dt \leq \left( \int_{\mathbb{R}^d} |\phi|^p dx \right)^{p-1} \int_{B(R)} \mathbb{E}(|\theta_n(t, x)|^p) dx dt$$

$$\leq C_p \left( \int_{\mathbb{R}^d} |\phi|^p dx \right)^{p-1} \int_{B(R)} (1 + |x|^p) dx.$$  (5.16)

where $q$ is the conjugate number of $p$. Let $M > 1$. We have

$$|\theta_0^p(X_t^{-1}(x)) - \theta_0(x)|^p = |\theta_0^p(X_t^{-1}(x)) - \theta_0(X_t^{-1}(x))|^p 1_{|X_t^{-1}(x)| \geq M}$$

$$+ |\theta_0^p(X_t^{-1}(x)) - \theta_0(X_t^{-1}(x))|^p 1_{|X_t^{-1}(x)| < M}.$$  (5.17)
The second term on the right side of (5.17) is dominated by
\[ \sup_{|y| \leq M} \left( \left| \theta_n^0(y) - \theta_0(y) \right|^2 \right) \rightarrow 0 \]
as \( n \rightarrow +\infty \). By (5.15), the expectation of the first term on the right side of (5.17) is dominated by \( C_1 \left( 1 + |x|^{2\omega} / M \right) \). Therefore,
\[ \sup_{|x| \leq M} \mathbb{E} \left( \left| \theta_n^0(t, x) - \theta(t, x) \right|^2 \right) \]
\[ \leq \frac{C(1 + R^{2\omega})}{M} + \sup_{|x| \leq M} \left| \theta_n^0(x) - \theta_0(x) \right|^2, \quad M > 1. \]

It follows that
\[ \lim_{n \rightarrow +\infty} \sup_{|x| \leq R} \mathbb{E} \left( \left| \theta_n(t, x) - \theta(t, x) \right|^2 \right) = 0 \]
which implies that for each \( \phi \in C_0^\infty(\mathbb{R}^d) \) with \( \text{supp} \phi \subset B(R) \)
\[ \mathbb{E} \left[ \left( (\theta_n(t), \phi)_{L^2} - (\theta(t), \phi)_{L^2} \right)^2 \right] \]
\[ \leq \left( \int_{\mathbb{R}^d} |\phi|^2 dx \right) \int_{B(R)} \mathbb{E}(\left| \theta_n^0(t, x) - \theta_0(t, x) \right|^2) dx \rightarrow 0. \quad (5.18) \]

Now (5.16) and (5.18) allows to pass to the limit so that (5.13) holds true in this case. \( \square \)

**Remark.** The DiPerna–Lion’s theory of transport equations with Sobolev space coefficients has been extended to stochastic context in [24].

**APPENDIX**

**Isotropic Flows on \( S^d \) Relative to the Critical Sobolev Exponent**

Sobolev isotropic flows have been discussed in [11]. The present section is motivated by [13]. Let \( S^d \) be the sphere of dimension \( d \geq 2 \) and \( \Delta \) the Laplace operator on vector fields over \( S^d \). It is well known that the space of vector fields is the direct sum of the space of gradient vector fields and the space of divergence free vector fields. Let \( c_{\ell,d} = (\ell + d - 2) \), \( c_{\ell,d} = (\ell + 1)(\ell + d - 2) \). Then spectrum(\( \Delta \)) = \( \{ -c_{\ell,d}; \ell \geq 1 \} \cup \{ -c_{1,d}; \ell \geq 1 \} \). Let \( \mathcal{G}_{\ell} \) be the eigenspace associated to \( c_{\ell,d} \) and \( \mathcal{D}_{\ell} \) the eigenspace associated to \( c_{\ell,d} \). Denote
\[ K_{\ell,1} = \dim \mathcal{G}_{\ell}, \quad K_{\ell,2} = \dim \mathcal{D}_{\ell}. \]
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It is known that

\[ K_{\ell,1} \sim O(\ell^{d-1}), \quad K_{\ell,2} \sim O(\ell^{d-1}) \quad \text{as} \quad \ell \to +\infty. \]

Let \( \{ A_{\ell,k}^i; k = 1, \ldots, K_{\ell,i}, \ell \geq 1 \} \) for \( i = 1, 2 \) be an orthonormal basis in \( L^2 \):

\[ \int_{S^d} \langle A_{\ell,k}^i(x), A_{\ell,k}^j(x) \rangle dx = \delta_{ij} \delta_{\ell k}. \]

By Weyl theorem, the vector fields \( \{ A_{\ell,k}^i \} \) are smooth.

Let \( s > 0 \) and \( H^s(S^d) \) be the Sobolev space of vector fields on \( S^d \), which is the completion of smooth vector fields relative to the norm

\[ \| V \|_{H^s}^2 = \int_{S^d} ((-\Delta + 1)V, V) dx. \]

Therefore,

\[ \{ A_{\ell,k}^i/(1 + c_{\ell,d})^{\ell/2}, A_{\ell,k}^j/(1 + c_{\ell,d})^{\ell/2} \} \]

is an orthonormal basis relative to \( H^s \). Let

\[ a_\ell = \frac{a}{\ell^{1+\alpha}}, \quad b_\ell = \frac{b}{\ell^{1+\alpha}}, \quad \alpha > 0, \quad a, b > 0. \]

We have

\[ \sqrt{a_\ell} K_{\ell,1} \sim O \left( \frac{1}{\ell^{(2+\alpha)/2}} \right), \quad \sqrt{b_\ell} K_{\ell,2} \sim O \left( \frac{1}{\ell^{(2+\alpha)/2}} \right). \] (6.1)

Let \( \{ B_{\ell,k}^i(t); \ell \geq 1, 1 \leq k \leq K_{\ell,i} \} \) for \( i = 1, 2 \) be two family of real independent standard Brownian motions defined on a probability space \((\Omega, \mathcal{F}, P)\). Consider the following Stratonovich SDE on \( S^d \):

\[ dx_t = \sum_{\ell \geq 1} \left\{ \frac{d a_\ell}{K_{\ell,1}} \sum_{k=1}^{K_{\ell,1}} A_{\ell,k}^i(x_t) \circ dB_{\ell,k}^i(t) + \frac{d b_\ell}{K_{\ell,2}} \sum_{k=1}^{K_{\ell,2}} A_{\ell,k}^j(x_t) \circ dB_{\ell,k}^j(t) \right\}. \] (6.2)

It was proved in [6] that for the critical exponent \( \alpha = 2 \), the SDE (6.2) defines a flow of homeomorphisms of \( S^d \). In what follows, we shall consider the case where \( a = 0 \). Let \( X_t(x) \) be the solution to

\[ dX_t = \sum_{\ell \geq 1} \left\{ \frac{d b_\ell}{K_{\ell,2}} \sum_{k=1}^{K_{\ell,2}} A_{\ell,k}^j(X_t) \circ dB_{\ell,k}^j(t) \right\}. \] (6.3)
Theorem 6.1. Let $\theta_0 \in C(S^d)$. Then $\theta(t, x) = \theta_0(X_t^{-1}(x))$ solves the following stochastic transport equation
\[
d\theta(t, x) + \sum_{\ell \geq 1} \left\{ \sqrt{\frac{d b_\ell}{K_{t, 2}}} \sum_{k=1}^{K_{t, 2}} \langle \nabla \theta(t, x), A^2_{\ell, k} \rangle \circ dB^2_{\ell, k}(t) \right\} = 0 \quad (6.4)
\]
in distribution sense, that is, for any $\phi \in C^\infty(\mathbb{S}^d)$,
\[
(\theta_t, \phi)_{L^2} = (\theta_0, \phi)_{L^2} - \sum_{\ell \geq 1} \left\{ \sqrt{\frac{d b_\ell}{K_{t, 2}}} \sum_{k=1}^{K_{t, 2}} \int_0^t \left( \theta_s, \text{div}(\phi A^2_{\ell, k}) \right) \circ dB^2_{\ell, k}(s) \right\}. \quad (6.5)
\]

Proof. Equation (6.5) admits the following Itô form
\[
(\theta_t, \phi)_{L^2} = (\theta_0, \phi)_{L^2} - \sum_{\ell \geq 1} \left\{ \sqrt{\frac{d b_\ell}{K_{t, 2}}} \sum_{k=1}^{K_{t, 2}} \int_0^t \left( \theta_s, \text{div}(\phi A^2_{\ell, k}) \right) \circ dB^2_{\ell, k}(s) \right\}
- \frac{1}{2} \sum_{\ell \geq 1} \left\{ \frac{d b_\ell}{K_{t, 2}} \sum_{k=1}^{K_{t, 2}} \int_0^t \left( \theta_s, \text{div}(\phi A^2_{\ell, k}) \right) ds \right\}. \quad (6.6)
\]

Since $\{A^2_{\ell, k}\}$ are divergence free, we have
\[
\text{div}(\phi A^2_{\ell, k}) = \left( A^2_{\ell, k}, \nabla \phi, A^2_{\ell, k} \right) = \left( \nabla \Phi_{\ell, k}, A^2_{\ell, k} \right) = \left( \nabla \Phi_{\ell, k}, A^2_{\ell, k} \right).
\]
It is known (see e.g., [6]) that
\[
\sum_{k=1}^{K_{t, 2}} \nabla \Phi_{\ell, k} A^2_{\ell, k} = 0 \quad \text{and} \quad \sum_{k=1}^{K_{t, 2}} \frac{d}{K_{t, 2}} \left[ \nabla \Phi_{\ell, k}, A^2_{\ell, k} \right] = \Delta \phi
\]
here $\Delta$ is the Laplace operator on functions on $S^d$. Therefore the last term in (6.6) is reduced to
\[
- \frac{1}{2} \sum_{\ell \geq 1} \left\{ \frac{d b_\ell}{K_{t, 2}} \int_0^t \left( \theta_s, \Delta \phi \right) ds \right\}.
\]

Now let $X^n_t$ be the solution to
\[
dX^n_t = \sum_{\ell = 1}^{2^n} \left\{ \sqrt{\frac{d b_\ell}{K_{t, 2}}} \sum_{k=1}^{K_{t, 2}} A^2_{\ell, k}(X^n_t) \circ dB^2_{\ell, k}(t) \right\}. \quad (6.7)
\]
and consider $\theta^n(t, x) = \theta_0((X^n_t)^{-1}(x))$. Then by what was discussed in Section 5, $\theta^n$ satisfies

$$
(\theta^n_t, \phi)_{L^2} = (\theta_0, \phi)_{L^2} - \sum_{\ell=1}^{2^n} \left\{ \int_0^t \frac{dB^{1,2}_\ell}{K_{1,2}} \sum_{k=1}^{K_{1,2}} \left( \theta^n_{\ell, k}, \text{div}(\phi A^n_{\ell, k}) \right)_{L^2} \cdot dB^{1,2}_\ell(s) \right\} - \frac{1}{2} \sum_{\ell=1}^{2^n} \left\{ b_{\ell, 1} \int_0^t \left( \theta^n_{\ell, 1}, \Delta \phi \right)_{L^2} ds \right\}.
$$

(6.8)

Again by [6], almost surely, $(X^n_t)^{-1}(x)$ converges to $X^{-1}_t(x)$ uniformly with respect to $x \in S^d$. Now letting $n \to +\infty$ in (6.8), we see that $\theta_t$ satisfies (6.6) or (6.5) holds. $\square$

REFERENCES


