Topological and $H$-Flux of $T$-Dual Manifolds

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We present a general formula for the topology and $H$-flux of the $T$-dual of a type II compactification. Our results apply to $T$-dualities with respect to any free circle action. In particular, we find that the manifolds on each side of the duality are circle bundles whose curvatures are given by the integral of the dual $H$-flux over the dual circle. As a corollary we conjecture an obstruction to multiple $T$-dualities, generalizing the obstruction known to exist on the twisted torus. Examples include SU(2) Wess-Zumino-Witten models, lens spaces, and the supersymmetric string theory on the nonspin AdS$^3 \times \mathbb{C}P^2 \times S^1$ compactification.

\[ f : S^1 \times E \to E, \quad \tilde{f} : \tilde{S}^1 \times \tilde{E} \to \tilde{E}. \] (1)

The spaces of orbits of these actions are 9-manifolds which we call $M$ and $\tilde{M}$. The freeness of the actions implies that each orbit is a loop and that none of these loops degenerate. As a result $E$ and $\tilde{E}$ are circle bundles over the bases $M$ and $\tilde{M}$, and so their topologies are entirely determined by the topology of their bases together with the curvatures $F, \tilde{F}$ (the first Chern classes of the bundles).

As we will see nontrivial bundles are $T$-dual to configurations with $H$-flux. Thus we will need to include the fluxes $H$ and $\tilde{H}$ in our two compactifications. The two configurations are then topologically determined by the triples $(M, F, H)$ and $(\tilde{M}, \tilde{F}, \tilde{H})$ where $M$ and $\tilde{M}$ are 9-manifolds, $F$ and $\tilde{F}$ are two-forms on $M$ and $\tilde{M}$, and $H$ and $\tilde{H}$ are three-forms on the total spaces $E$ and $\tilde{E}$. To capture the topology of a configuration it will suffice to consider the field strengths and Chern classes as elements of integral cohomology. This perspective is useful in that it automatically identifies some gauge equivalent configurations, excludes configurations not satisfying some equations of motion, and imposes the Dirac quantization conditions.

Our result.—The compactifications topologically specified by $(M, F, H)$ and $(\tilde{M}, \tilde{F}, \tilde{H})$ are $T$-dual if and only if

\[ M \cong \tilde{M}, \quad F = \int_{S^1} \tilde{H}, \quad \tilde{F} = \int_{S^1} H. \] (2)

This condition determines, at the level of cohomology, the curvatures $F$ and $\tilde{F}$. However, the NS field strengths are determined only up to the addition of a three-form on the base $M$, because the integral of such a form over the circle fiber vanishes. We further impose that the two dual three-forms on $M$ must be equal, as is made precise in Ref. [7], where the duality map of the RR fields (viewed both as elements of cohomology and of K-theory) can also be found.

In the remainder of this Letter we provide examples and applications of our result. When the curvatures $F$ and $\tilde{F}$ are topologically trivial (in the second cohomology of $M$) the bundles are trivial and so our two spacetimes are both topologically the trivial bundle $M \times S^1$. Using the K"unneth formula we may decompose the
third cohomology of the total space $M \times S^1$,

\[ H^3(M \times S^1) = H^3(M) \otimes H^0(S^1) \oplus H^2(M) \otimes H^1(S^1) = H^3(M) \oplus H^2(M), \tag{3} \]

and so the NS fluxes $H$ and $\tilde{H}$, being elements of $H^3(M \times S^1)$, decompose as $H = \alpha + \beta d\theta$, $\tilde{H} = \tilde{\alpha} + \beta d\theta$, where $\alpha, \tilde{\alpha} \in H^3(M)$, $\beta, \tilde{\beta} \in H^2(M)$, and $d\theta$ is the generator of $H^1(S^1) = \mathbb{Z}$. Integrating $H$ and $\tilde{H}$ over the circle, using the normalization $\int d\theta = 1$, our result yields $\alpha = \tilde{\alpha}$, $\beta = \tilde{\beta} = 0$. Thus we reproduce the original examples of $T$-duality, in which spacetime is the product of a 9-manifold and a circle and the $H$-flux is an element of the cohomology of the 9-manifold. As expected the $T$-dual is also a product manifold and carries the same $H$-flux.

The next most trivial case is a trivial circle bundle with $H$-flux, which we see from (2) is $T$-dual to a nontrivial bundle without $H$-flux. In this case our result was demonstrated using $S$-duality and also using the $E_8$ gauge bundle formalism in Ref. [7].

The simplest nontrivial circle bundle is the Hopf fibration over the 2-sphere. This bundle is constructed by cutting the 2-sphere into a northern $S^2_N$ and southern hemisphere $S^2_S$, over which the circle is trivially fibered. The two hemispheres are then glued together along the equator. In particular, each point on the equator is specified by a longitude $\theta$ and the attaching map identifies the point $\phi$ on the fiber over one hemisphere to the point $\phi + \theta$ on the other. The total space of this bundle is then the three-sphere $S^3$ and the curvature $F$ is the generator [1] of $H^2(S^2) = \mathbb{Z}$.

Now we may consider a type II string theory compactification on a 3-sphere crossed with an irrelevant 7-manifold, express the 3-sphere as the circle bundle above, and then $T$-dualize with respect to the circle fiber.

Let us begin with a case in which there is no $H$-flux, such as a type IIB compactification of $AdS^3 \times T^4 \times S^3$ supported by RR flux. Applying Eq. (2) with $F = [1] \in H^2(T^3)$ and $H = 0$, we find $\tilde{F} = [0]$ and $\tilde{H} = [1] \in H^3(S^3 \times S^1)$. Thus our nontrivial bundle $S^3$ becomes the trivial bundle $S^2 \times S^1$, supported by one unit of $H$-flux. Incidentally the construction in Ref. [7] tells us that if there was RR 3-form flux $G_3 = [k] \in H^3(S^3)$ on the $S^3$, then we would find $\tilde{G}_3 = [k] \in H^2(S^3)$, whereas the $G_3$-flux on the $AdS^3$ becomes $G_3$-flux on $AdS^3 \times S^1$. This is in accord with the usual intuition in which $T$-duality toggles whether an RR flux extends along the circle, except that here we see that this intuition may apply even when the circle is nontrivially fibered. One might think that this would be impossible in general because, for example, here, there is no first cohomology class corresponding to the circle in the 3-sphere. This is potentially problematic because, for example, an arbitrary integral Romans mass $G_0 \in H^0(S^2 \times S^3)$ cannot be dual to an element $G_1 \in H^1(S^3)$ since $H^1(S^3) = 0$. Here we are saved from any contradiction by the supergravity equation of motion $\tilde{G}_0 H = 0$, which implies that $\tilde{G}_0 = [0]$. More generally we are saved by the quantum corrected equations of motion, which are given by the Freed-Witten anomaly [11].

A famous application of the previous example is the $T$-duality of string theory on $\mathbb{R}^{k,1} \times S^1$ with an NS5-brane extended along the plane $\mathbb{R}^{k,1} \subset \mathbb{R}^{k,1}$ and localized at a point $\theta \in S^1$. Such an NS5-brane is linked by an $S^2 \times S^3$ where $S^2 \subset \mathbb{R}^{k,1}$ links the $\mathbb{R}^{k,1}$ plane. Recalling that NS5-branes are magnetic sources for $NS$ flux, Gauss's law allows us to integrate \( \int_{S^2 \times S^3} H = 1 \) and so

\[ H = [1] \in H^3(S^2 \times S^1) = \mathbb{Z}. \tag{4} \]

As we have seen, a $T$-duality along this circle replaces $S^2 \times S^1$ with a 3-sphere and the $H$-flux disappears. As the $H$-flux has disappeared the $T$-dual compactification has no NS5-brane; instead it has been replaced by a circle bundle which is nontrivially fibered over each sphere linking the 6-submanifold where the NS5-brane was. This submanifold is now a Kaluza-Klein (KK) monopole for the dual U(1). Thus we have recovered the familiar fact that NS5-branes are $T$-dual to KK monopoles (see, e.g., [12], and references therein). If, instead of a single NS5-brane, we had considered a stack of $k$ NS5-branes, then we would have had $H = [k]$ and so the dual bundle would again have been nontrivial, this time yielding a KK-monopole charge of $k$. More generally we may consider nontrivial circle bundles and nontrivial $H$-flux at the same time. For example, string theory on the Lens space $S^3/\mathbb{Z}_j$ with $k$ units of $H$-flux is $T$-dual to string theory on $S^3/\mathbb{Z}_j$ with $j$ units of $H$-flux [13].

An example of $T$-duality that has recently received attention in the literature [5] is the duality between circle bundles over a 2-torus $T^2$. In particular, one may start with a 3-torus, which is the trivial circle bundle over a 2-torus, with $k \neq 0$ units of $H$-flux and then $T$-dualize with respect to the circle fiber. As above, $F = [0] \in H^2(T^3) = \mathbb{Z}$ and $H = [k] \in H^1(T^3) = \mathbb{Z}$ determine the dual curvature and $NS$ flux

\[ \tilde{F} = [k] \in H^2(T^2) = \mathbb{Z}, \quad \tilde{H} = [0] \in H^1(\tilde{E}) = \mathbb{Z}. \tag{5} \]

Here the dual manifold $\tilde{E}$, commonly referred to as a twisted torus or nilmanifold, is the circle bundle over $T^2$ characterized entirely by the curvature $\tilde{F} = [k]$.

We may also try to $T$-dualize a larger subtorus of the original $T^3$. This means that after $T$-dualizing with respect to the fiber circle we may then try to $T$-dualize with respect to one of the circles in the base. As has been found in Ref. [5], this is impossible. In particular, after the first circle is $T$-dualized, the other two circles have ceased to be globally defined and so cannot be $T$-dualized. Had they both been globally defined we could have defined a nowhere-vanishing section of this nontrivial circle bundle. Thus in general we see that it is impossible to $T$-dualize with respect to any 3-torus supporting $H$-flux. In fact, the example in Ref. [5] suggests the stronger result
that one cannot $T$-dualize on a 2-torus, unless $\int_T H = 0$

A critical check of any proposed duality is that the
anomalies match on both sides. Although a more general
matching of a particular gravitino anomaly was demon-
strated in Ref. [7], here we describe one family of ex-
amples which illustrates the general pattern. Consider the
famous type IIB string theory compactification on
$\text{AdS}^3 \times S^5$, where $\text{AdS}^3$ is the five-dimensional an-
de-Sitter space. The 5-sphere $S^5$ is a circle bundle over
the complex projective plane $\mathbb{CP}^2$ with a single unit of
curvature $F = [1] \in H^2(\text{AdS}^3 \times \mathbb{CP}^2) = H^2(\mathbb{CP}^2) = \mathbb{Z}$.

The second equality is a result of the fact that $\text{AdS}^3$ is
contractible and so it does not contribute to the cohomol-
y groups; we may thus freely omit it from such equa-
tions. The third cohomology group of this spacetime is
trivial and so the $H$-flux must be topologically trivial.

In addition, there is a RR 5-form field strength $G_5 = [N] \in
H^4(S^5)$. There is also $G_5$-flux along the noncompact
directions.

$T$-dualizing with respect to the fiber circle we find that
the dual curvature $\tilde{F}$ vanishes. The dual manifold is thus
$\text{AdS}^3 \times \mathbb{CP}^2 \times S^1$. Equation (2) is satisfied only if the
dual NS field strength is
\[
\tilde{H} = [1] \in H^3(\text{AdS}^3 \times \mathbb{CP}^2 \times S^1) = H^2(\mathbb{CP}^2) \otimes H^1(S^1) = \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}.
\]

The $G_5$-flux becomes $\tilde{G}_4 = [N] \in H^4(\mathbb{CP}^2) = \mathbb{Z}$ and $\tilde{G}_6$ is
supported in the noncompact directions crossed with the
circle. In particular, there is no $\tilde{G}_4$-flux and therefore
the M-theory circle is trivially fibered, meaning that this
configuration is dual to M-theory compactified on
$\text{AdS}^3 \times \mathbb{CP}^2 \times T^2$. Notice that $\mathbb{CP}^2$ is not spin, and so
this is an M-theory compactification on a nonspin mani-
fold. This might seem impossible, because the low energy
effective supergravity has a gravitino and so there must be
a spin structure. But, in fact, this compactification was
trivial and so the $\text{AdS}^3$ configuration is dual to M-
theor y compactified on a nonspin manifold.

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