Harnack Inequalities for SDEs with Hölder Continuous Drift

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Abstract

Harnack inequalities for stochastic differential equations with non-degenerated diffusion coefficient and Hölder continuous drift coefficient are established. To this end, we will adopt a special Itô–Tanaka type transformation of the drift developed in [4]. Moreover, for non-degenerate SDEs with singular time-dependent drift coefficient studied in [6, 18], we establish the log-Harnack inequality based on the gradient estimate and semigroup method. Finally, by using explicit heat kernel estimates for stable processes with drift, we also prove Harnack inequalities for stochastic differential equations driven by symmetric stable processes.

Keywords: Harnack inequality, Hölder continuity, Itô–Tanaka type transformation

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1 Introduction and Main Result

The dimension-free Harnack inequality with powers introduced in [11] and the log-Harnack inequality introduced in [8] have been intensively investigated for various stochastic (partial) differential equations. They are efficiently applied to study heat kernel estimates, functional inequalities, transportation-cost inequalities and properties of invariant measures, see e.g. [13] and references therein. Consider the following stochastic differential equation (SDE) on $\mathbb{R}^d$:

$$dX_t = \sigma(X_t) \, dW_t + b(X_t) \, dt, \quad X_0 = x,$$

where $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ are two Borel measurable functions, and $\{W_t\}_{t \geq 0}$ is a standard $d$-dimensional Brownian motion. When the equation (1.1) has a unique solution for any starting point $x$, we denote it by $X_t(x)$ and define the associated Markov semigroup $(P_t)_{t \geq 0}$ as follows:

$$P_t f(x) = E f(X_t(x)), \quad t \geq 0, x \in \mathbb{R}^d, f \in B_b(\mathbb{R}^d).$$

If the coefficients $\sigma$ and $b$ are globally semi-Lipschitz continuous, Harnack inequalities for $P_t$ have been established in [12]; see Theorem 2.3 below for the explicit statement. Recently, Harnack inequalities for (1.1) with log-Lipschitz continuous coefficients have been studied in [10]. The
aim of this paper is to consider Harnack inequalities for SDE (1.1) with Hölder continuous drift coefficient.

For this purpose, we mainly adopt the framework of [5]. First, we recall the following assumptions used there (see Subsection 2.1 for the definitions of the function spaces).

(H1) There is a constant $\theta \in (0, 1)$ such that $b \in C^0(\mathbb{R}^d, \mathbb{R}^d)$.

(H2) The diffusion coefficient $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a bounded measurable function belonging to $C^0(\mathbb{R}^d, \mathbb{R}^d \otimes \mathbb{R}^d)$.

(H3) For any $x \in \mathbb{R}^d$, the inverse of $a(x) := \sigma(x)\sigma(x)^*$ exists and

$$\|a^{-1}\|_0 := \sup_{x \in \mathbb{R}^d} \|a^{-1}(x)\|_{HS} < +\infty,$$

where $\|a^{-1}(x)\|_{HS}$ is the Hilbert–Schmidt norm of the matrix $a^{-1}(x)$.

According to [5, Theorem 7 (i)], the equation (1.1) has a unique strong solution $X_t$ which forms a stochastic flow of diffeomorphisms on $\mathbb{R}^d$. We shall prove

**Theorem 1.1.** Assume the hypotheses (H1)–(H3). Let $(P_t)_{t \geq 0}$ be the semigroup associated to the Itô SDE (1.1). Then there are three positive constants $K$, $\kappa$ and $\delta$ such that

(1) for any $T > 0$ and $f \in \mathcal{B}_b^+(\mathbb{R}^d)$ with $f \geq 1$, it holds

$$P_T \log f(y) \leq \log P_T f(x) + \frac{2K|x-y|^2}{\kappa^2(1 - e^{-Kt})} \text{ for all } x, y \in \mathbb{R}^d; \quad (1.2)$$

(2) for $p > (1 + \delta/\kappa)^2$ and $\delta_p := \max\{\delta, \kappa(\sqrt{p} - 1)/2\}$, it holds

$$(P_T f(y))^p \leq (P_T f^p(x)) \exp \left[ \frac{K\sqrt{p}(\sqrt{p} - 1)|x-y|^2}{\delta_p(\sqrt{p} - 1)\kappa - \delta_p(1 - e^{-Kt})} \right] \quad (1.3)$$

for all $f \in \mathcal{B}_b^+(\mathbb{R}^d), x, y \in \mathbb{R}^d$ and $T > 0$.

Unfortunately, the explicit expressions of the constants $K$, $\kappa$ and $\delta$ are a little complicated, as can be seen from the proof in the next section. To point out the main difference between the Hölder continuous situation and the Lipschitz continuous setting, we first explain the idea of coupling for (1.1) with semi-Lipschitz continuous drift. For simplicity, we consider the easy case where $\sigma = \text{Id}$ and for some $K \in \mathbb{R}$,

$$\langle b(x) - b(y), x - y \rangle \leq K|x - y|^2, \quad x, y \in \mathbb{R}^d.$$

For $x \neq y \in \mathbb{R}^d$ and $T > 0$, let $X_t$ solve (1.1) with $X_0 = x$, and $Y_t$ solve

$$dY_t = dB_t + b(Y_t) \, dt + \frac{(X_t - Y_t)|x - y|e^{-Kt}}{|X_t - Y_t|} \int_0^T e^{-2Ks} \, ds, \quad Y_0 = y.$$

Then, $Y_t$ is well defined up to the coupling time

$$\tau = \inf\{t \geq 0 : X_t = Y_t\}.$$

Let $Y_t = X_t$ for $t \geq \tau$. We have

$$d|X_t - Y_t| \leq K|X_t - Y_t|dt - \frac{|x - y|e^{-Kt}}{\int_0^T e^{-2Ks} \, ds} dt, \quad t \leq \tau.$$
That is,
\[ d(|X_t - Y_t|e^{-Kt}) \leq \frac{|x - y|e^{-2Kt}}{\int_0^t e^{-2Ks} \, ds} \, dt, \quad t \leq \tau. \]

This implies \( \tau \leq T \) and hence, \( X_T = Y_T \). Combining it with the Girsanov theorem yields the desired Harnack inequalities; see for instance the proof of [1, Theorem 2] or that of [12, Theorem 1.1]. However, due to the poor Hölder regularity of the drift vector field \( b \), it seems that in the present setting one cannot directly use the coupling method above to establish the Harnack inequalities.

Now we briefly describe our strategy to help the readers understand the proof of Harnack inequalities with Hölder continuous drift better. Following the ideas in the proof of [5, Theorem 7], we can transform the equation (1.1) into a new SDE (2.3) which has smooth coefficients with bounded derivatives; moreover, there is a simple relationship between their corresponding semigroups (see (2.4) below). For this new equation (2.3), we can check that the assumptions (A1)–(A3) in [12] are satisfied under our hypotheses (H1)–(H3). In this way we first get Harnack inequalities for the semigroup associated with the new equation (2.3), then the relationship (2.4) between the semigroups allows us to prove Theorem 1.1.

J. Shao established in [9, Theorem 2.1] the Harnack inequalities for SDEs with singular drifts satisfying an integrability condition. In the inequalities (2.3) and (2.4) of [9], there are some extra constants on the right hand sides. We are unable to remove the constant in the Harnack inequality [9, (2.4)] by directly using our method. The reason is that the diffusion coefficient of the new SDE obtained by using the Zvonkin transform is only Hölder continuous (cf. [17, (8) and (11)]). Nevertheless, we can improve the log-Harnack inequality [9, (2.3)], thanks to the \( L^2 \)-gradient estimate given in the proof of [18, Theorem 3.5] (see also (3.7) of the current paper).

This paper is organized as follows. In Section 2, we first recall some necessary results from our main references [5, 12], then the main part is devoted to check that the coefficients of the transformed SDE (2.3) verify the hypotheses (A1)–(A3) in [12]. With the key relation (2.4) in hand, it is easy to give the proof of Theorem 1.1. Based on the \( L^2 \)-gradient estimate in [18, p.1109], we establish in Section 3 the log-Harnack inequality by applying the semigroup method and the Zvonkin transformation. Finally, by using explicit heat kernel estimates, we establish in Section 4 the Harnack inequality for SDEs driven by \( \alpha \)-stable process.

2 Proof of Theorem 1.1

This section is divided into three parts. In subsections 2.1 and 2.2, we first give some preliminary results and we shall prove Theorem 1.1 in the last subsection.

2.1 Flandoli–Gubinelli–Priola’s Transformation

We begin with some notations about the space of Hölder continuous functions which are the same as those in [5]. For \( d, k \geq 1 \) and \( \theta \in (0, 1) \), we write \( C^\theta(\mathbb{R}^d, \mathbb{R}^k) \) for the space of functions \( f : \mathbb{R}^d \rightarrow \mathbb{R}^k \) satisfying

\[ [f]_{\theta} := \sup_{x \neq y \in \mathbb{R}^d, |x - y| \leq 1} \frac{|f(x) - f(y)|}{|x - y|^{\theta}} < +\infty. \]

These are locally uniformly \( \theta \)-Hölder continuous functions. Let

\[ [f]_{\theta, 1} := \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|^\theta \vee |x - y|}. \]
where $a \vee b = \max(a, b)$, $a, b \in \mathbb{R}$. It is easy to know that $|f|_\theta \leq |f|_{\theta, 1} \leq 2|f|_\theta$, thus the functions in $C^\theta(\mathbb{R}^d, \mathbb{R}^k)$ have at most linear growth. When endowed with the norm

$$
\|f\|_\theta = \|(1 + | \cdot |)^{-1} f(\cdot)\|_0 + |f|_\theta,
$$

where $\|\cdot\|_0$ is the usual supremum norm, the set $C^\theta(\mathbb{R}^d, \mathbb{R}^k)$ becomes a Banach space. A function $f$ is said to belong to $C^{n+\theta}(\mathbb{R}^d, \mathbb{R}^k)$, $n \geq 1$, if it is continuously differentiable up to order $n$ and $D^n f$ is $\theta$-Hölder continuous. Define the corresponding norm by

$$
\|f\|_{n+\theta} = \|(1 + | \cdot |)^{-1} f(\cdot)\|_0 + \sum_{i=1}^{n} \|D^i f\|_0 + \|D^n f\|_\theta.
$$

If $k = 1$, $C^{n+\theta}(\mathbb{R}^d, \mathbb{R})$ is simply denoted by $C^{n+\theta}(\mathbb{R}^d)$, $n \geq 0$. We write $C^{n+\theta}_b(\mathbb{R}^d, \mathbb{R}^k)$ for the subspace of bounded functions in $C^{n+\theta}(\mathbb{R}^d, \mathbb{R}^k)$. In particular, $C^{0}_b(\mathbb{R}^d, \mathbb{R}^k)$ is the space of all bounded functions $f : \mathbb{R}^d \to \mathbb{R}^k$ having bounded derivatives up to order $n$.

Denote by $\mathcal{L}$ the generator of the solution $X_t$ to the equation (1.1), i.e., for $f \in C^2(\mathbb{R}^d),$

$$
\mathcal{L} f(x) = \frac{1}{2} \text{Tr}[a(x)D^2 f(x)] + \langle b(x), D f(x) \rangle.
$$

The following regularity result is taken from [5, Theorem 5] (see also [4, Lemma 4]).

**Lemma 2.1.** Assume the conditions (H1)–(H3). For any $\theta' \in (0, \theta)$, there exists a constant $\lambda_0 > 0$ (depending on $\theta, \theta', d, |\sigma|_0, |a^{-1}|_0$ and $\|D^k \sigma\|_0, k = 1, 2, 3$) such that for any $\lambda \geq \lambda_0$ and $f \in C^\theta(\mathbb{R}^d)$, the equation

$$
\lambda u - \mathcal{L} u = f
$$

has a unique classical solution $u = u_\lambda \in C^{2+\theta'}(\mathbb{R}^d)$ for which

$$
\|u\|_{2+\theta'} = \|u(\cdot)(1 + | \cdot |)^{-1}\|_0 + \|Du\|_0 + \|D^2 u\|_0 + \|D^2 u\|_{\theta'} \leq C(\lambda)\|f\|_\theta,
$$

where $C(\lambda)$ (independent of $u$ and $f$) satisfies $C(\lambda) \to 0$ as $\lambda \to \infty$.

Now for given $\lambda > 0$ we consider the equation

$$
\lambda \psi_\lambda - \mathcal{L} \psi_\lambda = b.
$$

(2.1)

By Lemma 2.1, we know that when $\lambda > \lambda_0$, there exists a classical solution $\psi_\lambda \in C^{2+\theta'}(\mathbb{R}^d, \mathbb{R}^d)$. Define $\Psi_\lambda(x) = x + \psi_\lambda(x), x \in \mathbb{R}^d$. Then we have (see [5, Lemma 8] or [4, Lemma 6])

**Lemma 2.2.** Assume (H1)–(H3). For $\lambda$ large enough, such that $\|D \psi_\lambda\|_0 < 1$, the following statements hold:

(i) $\Psi_\lambda$ has bounded first and second order derivatives and, moreover, the second order derivative $D^2 \Psi_\lambda$ is globally $\theta'$-Hölder continuous.

(ii) $\Psi_\lambda$ is a $C^2$-diffeomorphism of $\mathbb{R}^d$.

(iii) $\Psi_\lambda^{-1}$ has bounded first and second order derivatives and

$$
D \Psi_\lambda^{-1}(y) = \sum_{k \geq 0} \left[ - D \psi_\lambda(\Psi_\lambda^{-1}(y)) \right]^k.
$$
In the following, we will fix $\lambda$ large enough so that

$$\|D\psi\|_0 \leq 1/2.$$ 

We shall simply write $\psi$ and $\Psi$ for $\psi_\lambda$ and $\Psi_\lambda$, respectively. As in step 2 in the proof of [5, Theorem 7], we define

$$\tilde{\sigma}(y) = D\Psi(\Psi^{-1}(y))\sigma(\Psi^{-1}(y)) \quad \text{and} \quad \tilde{b}(y) = \lambda \psi(\Psi^{-1}(y)), \quad y \in \mathbb{R}^d. \quad (2.2)$$

Consider the following conjugated SDE:

$$d\tilde{X}_t = \tilde{\sigma}(\tilde{X}_t)\,dW_t + \tilde{b}(\tilde{X}_t)\,dt, \quad \tilde{X}_0 = y. \quad (2.3)$$

From Lemmas 2.1 and 2.2 we easily see that the functions $\tilde{\sigma} : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $\tilde{b} : \mathbb{R}^d \to \mathbb{R}^d$ are of class $C^{1+\theta'}$ for any $\theta' \in (0, \theta)$. In particular, they are Lipschitz continuous (see also the proof of Lemma 2.4(1) below). Hence, the equation (2.3) has a unique strong solution. Using the Itô formula, it is not difficult to see that (2.3) is equivalent to (1.1) in the following sense: if $X_t$ is a solution to (1.1), then $\tilde{X}_t = \Psi(X_t)$ satisfies (2.3) with $y = \Psi(x)$; conversely, if $\tilde{X}_t$ is a solution to (2.3), then $X_t = \Psi^{-1}(\tilde{X}_t)$ solves (1.1) with $x = \Psi^{-1}(y)$.

Denote by $P_t$ and $\tilde{P}_t$ the semigroups associated to the SDE (1.1) and (2.3), respectively. For any $f \in \mathcal{B}_b(\mathbb{R}^d)$, $g := f \circ \Psi^{-1} \in \mathcal{B}_b(\mathbb{R}^d)$, and

$$P_tf(x) = \mathbb{E}f(X_t(x)) = \mathbb{E}[(f \circ \Psi^{-1})(\Psi(X_t(x)))]$$

$$= \mathbb{E}g[\tilde{X}_t(\Psi(x))] = \tilde{P}_tg(\Psi(x)). \quad (2.4)$$

This is the key ingredient of the proof of Theorem 1.1.

### 2.2 Wang’s Harnack Inequalities for SDEs under semi-Lipschitz Condition

We first list the following assumptions (cf. [12, Introduction]):

(A1) there exists a constant $K_0 > 0$ such that

$$||\sigma(x) - \sigma(y)||_H^2 + 2\langle b(x) - b(y), x - y \rangle \leq K_0|x - y|^2, \quad x, y \in \mathbb{R}^d;$$

(A2) there is a constant $\kappa_0 > 0$ such that

$$\alpha(x) = \sigma(x)\sigma(x)^* \geq \kappa_0^2 \text{Id}, \quad x \in \mathbb{R}^d;$$

(A3) there is a constant $\delta_0 \geq 0$ such that

$$|\langle \sigma(x) - \sigma(y), x - y \rangle| \leq \delta_0|x - y|, \quad x, y \in \mathbb{R}^d.$$

It is well known that assumption (A1) ensures the pathwise uniqueness of solutions to (1.1). For the moment we assume that the SDE (1.1) has a unique strong solution $X_t$ and denote by $P_t$ the associated semigroup. F.-Y. Wang has shown in [12, Theorem 1.1] the following results on the Harnack inequalities for the semigroup $P_t$.

**Theorem 2.3.** (1) If (A1) and (A2) hold, then

$$P_T \log f(y) \leq \log P_T f(x) + \frac{K_0|x - y|^2}{2\kappa_0^2(1 - e^{-K_0T})}, \quad f \in \mathcal{B}_b^+(\mathbb{R}^d) \text{ with } f \geq 1, x, y \in \mathbb{R}^d, T > 0.$$

(2) If (A1), (A2) and (A3) hold, then for $p > (1 + \delta_0/\kappa_0)^2$ and $\delta_p := \max\{\delta_0, \kappa_0(\sqrt{p} - 1)/2\}$,

$$(P_T f(y))^p \leq (P_T f^p(x)) \exp \left[\frac{K_0\sqrt{p}(\sqrt{p} - 1)|x - y|^2}{4\delta_p((\sqrt{p} - 1)\kappa_0 - \delta_p)(1 - e^{-K_0T})}\right],$$

holds for all $f \in \mathcal{B}_b^+(\mathbb{R}^d)$, $x, y \in \mathbb{R}^d$ and $T > 0.$
2.3 Proof

In view of (2.4), to establish Harnack inequalities for the semigroup $P_t$, one can first study the semigroup $\tilde{P}_t$ associated to the SDE (2.3). For this aim, we need to check that $\tilde{\sigma}$ and $\tilde{b}$ satisfy the assumptions in Theorem 2.3.

**Lemma 2.4.** Under the hypotheses (H1)–(H3), the coefficients $\tilde{\sigma}$ and $\tilde{b}$ given by (2.2) satisfy the assumptions (A1)–(A3) in Theorem 2.3. More precisely, there exist positive constants $K_1, \kappa_1$ and $\delta_1$ such that for all $x, y \in \mathbb{R}^d$,

1. $\|\tilde{\sigma}(x) - \tilde{\sigma}(y)\|_{HS}^2 + 2(\tilde{b}(x) - \tilde{b}(y), x - y) \leq K_1|x - y|^2$;
2. $\tilde{a}(x) = \tilde{\sigma}(x)\tilde{\sigma}(x)^* \geq \kappa_1^2 \text{Id}$;
3. $|(\tilde{\sigma}(x) - \tilde{\sigma}(y))(x - y)| \leq \delta_1|x - y|$.

**Proof.** (1) For $x, y \in \mathbb{R}^d$,

$$
\|\tilde{\sigma}(x) - \tilde{\sigma}(y)\|_{HS} = \|D\Psi(\Psi^{-1}(x))\sigma(\Psi^{-1}(x)) - D\Psi(\Psi^{-1}(y))\sigma(\Psi^{-1}(y))\|_{HS}
\leq \|D\Psi(\Psi^{-1}(x)) - D\Psi(\Psi^{-1}(y))\|_{HS}\|\sigma(\Psi^{-1}(x)) - \sigma(\Psi^{-1}(y))\|_{HS}
+ \|D\Psi(\Psi^{-1}(y))\|_{HS}\|\sigma(\Psi^{-1}(x)) - \sigma(\Psi^{-1}(y))\|_{HS}.
$$

Since $\Psi(x) = x + \psi(x)$, we have $D\Psi(x) = \text{Id} + D\psi(x)$, and hence

$$
\|D\Psi(\Psi^{-1}(x)) - D\Psi(\Psi^{-1}(y))\|_{HS} = \|D\psi(\Psi^{-1}(x)) - D\psi(\Psi^{-1}(y))\|_{HS}
\leq \|D^2\psi\|_{0}\|\Psi^{-1}(x) - \Psi^{-1}(y)\|_{0}
\leq \|D^2\psi\|_{0}\|\Psi^{-1}\|_{0}|x - y|.
$$

As $\|D\psi\|_{0} \leq 1/2$, Lemma 2.2(iii) yields that $\|D\Psi^{-1}\|_{0} \leq \sum_{k \geq 0}\|D\psi\|^{k}_{0} \leq 2$. Thus,

$$
\|D\Psi(\Psi^{-1}(x)) - D\Psi(\Psi^{-1}(y))\|_{HS} \leq 2\|D^2\psi\|_{0}|x - y|.
$$

Next, it holds that

$$
\|\sigma(\Psi^{-1}(x)) - \sigma(\Psi^{-1}(y))\|_{HS} \leq \|D\sigma\|_{0}\|\Psi^{-1}(x) - \Psi^{-1}(y)\|_{0}
\leq \|D\sigma\|_{0}\|\Psi^{-1}\|_{0}|x - y|
\leq 2\|D\sigma\|_{0}|x - y|.
$$

Therefore, we have

$$
\|\tilde{\sigma}(x) - \tilde{\sigma}(y)\|_{HS} \leq 2\|D^2\psi\|_{0}\|\sigma\|_{0}|x - y| + 2\|D\Psi\|_{0}\|\sigma\|_{0}|x - y|
= 2\|D^2\psi\|_{0}\|\sigma\|_{0}|x - y| + 2\|D\Psi\|_{0}\|\sigma\|_{0}|x - y|.
$$

On the other hand, by (2.2) and the fact $\|D\Psi\|_{0}\|D\Psi^{-1}\|_{0} \leq (1/2) \times 2 = 1$,

$$
|\tilde{b}(x) - \tilde{b}(y), x - y| \leq |\tilde{b}(x) - \tilde{b}(y)||x - y|
\leq \lambda\|D\psi\|_{0}\|\Psi^{-1}(x) - \Psi^{-1}(y)\|_{0}|x - y|
\leq \lambda\|D\psi\|_{0}\|D\Psi^{-1}\|_{0}|x - y|^2
\leq \lambda|x - y|^2.
$$

Combining with all the estimates above, we get the desired estimate (A1) with

$$
K_1 = 4\|D^2\psi\|_{0}\|\sigma\|_{0} + \|D\Psi\|_{0}\|\sigma\|_{0} + 2\lambda.
$$
(2) For any \( x, z \in \mathbb{R}^d \), it follows from the definition of \( \tilde{\sigma} \) that
\[
\langle \tilde{\sigma}(x)\tilde{\sigma}(x)^* z, z \rangle = \langle D\Psi(\Psi^{-1}(x)) \sigma(\Psi^{-1}(x)) \sigma(\Psi^{-1}(x))^* [D\Psi(\Psi^{-1}(x))]^* z, z \rangle \\
= \langle a(\Psi^{-1}(x)) [D\Psi(\Psi^{-1}(x))]^* z, [D\Psi(\Psi^{-1}(x))]^* z \rangle.
\]
Denote by \( \lambda_i(x), 1 \leq i \leq d \), the eigenvalues of \( a(x) \). Under the hypothesis \((H3)\), we have \( \lambda_i(x) > 0 \) for all \( i \in \{1, \ldots, d\} \) and \( x \in \mathbb{R}^d \). By Cauchy’s inequality,
\[
\sum_{i=1}^{d} \frac{1}{\lambda_i(x)} = \text{Tr}(a^{-1}(x)) \leq \sqrt{d} \left( \sum_{i=1}^{d} (a_{ii}^{-1}(x))^2 \right)^{\frac{1}{2}} \leq \sqrt{d} \|a^{-1}(x)\|_{HS} \leq \sqrt{d} \|a^{-1}\|_0.
\]
Hence
\[
\inf_{x \in \mathbb{R}^d} \lambda_i(x) \geq \frac{1}{\sqrt{d} \|a^{-1}\|_0} > 0, \quad 1 \leq i \leq d.
\]
As a consequence,
\[
\langle \tilde{\sigma}(x)\tilde{\sigma}(x)^* z, z \rangle \geq \frac{1}{\sqrt{d} \|a^{-1}\|_0^2} \|D\Psi(\Psi^{-1}(x))\|_{HS}^2 z^2.
\]
Noting that \( D\Psi = \text{Id} + D\psi \), for any \( y \in \mathbb{R}^d \),
\[
\|D\Psi(y)\|_{HS}^2 |z| = |z + [D\psi(y)]^* z| \geq |z| = [D\psi(y)]^* z \geq |z| = \|D\psi(y)\|_{HS} |z| \geq |z|/2,
\]
thanks to \( \|D\psi\|_0 = \sup_{y \in \mathbb{R}^d} \|D\psi(y)\|_{HS} \leq 1/2 \). Therefore
\[
\langle \tilde{\sigma}(x)\tilde{\sigma}(x)^* z, z \rangle \geq \frac{|z|^2}{4 \sqrt{d} \|a^{-1}\|_0^2},
\]
which means that the condition \((A2)\) holds with \( \kappa_1 = (4 \sqrt{d} \|a^{-1}\|_0^2)^{-1/2} \).

(3) Since \( \sigma \) is bounded, it is obvious that for any \( x \in \mathbb{R}^d \),
\[
\|\tilde{\sigma}(x)\|_{HS} \leq \|D\Psi(\Psi^{-1}(x))\|_{HS} \|\sigma(\Psi^{-1}(x))\|_{HS} \leq (d + \|D\psi\|_0) \|\sigma\|_0 \leq (d + 1/2) \|\sigma\|_0.
\]
Then the condition \((A3)\) holds with \( \delta_1 = (2d + 1) \|\sigma\|_0 \).

Now we are ready to prove Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 2.4 and Theorem 2.3, we know that the semigroup \( \tilde{P}_t \) corresponding to the SDE \((2.3)\) satisfies the log-Harnack inequality:
\[
\tilde{P}_T \log g(y) \leq \log \tilde{P}_T g(x) + \frac{K_1 |x - y|^2}{2K_1^2 (1 - e^{-K_{1}T})}, \quad T > 0, g \in B^+_E(\mathbb{R}^d) \text{ with } g \geq 1, x, y \in \mathbb{R}^d \quad (2.5)
\]
and the Harnack inequality with power: for \( p > (1 + \delta_1/\kappa_1)^2 \) and \( \delta_p := \max\{\delta_1, \kappa_1(\sqrt{p} - 1)/2\} \), it holds
\[
(\tilde{P}_T g(y))^p \leq (\tilde{P}_T g^p(x)) \exp \left[ \frac{K_1 \sqrt{p(\sqrt{p} - 1)|x - y|^2}}{4\delta_p [(\sqrt{p} - 1)\kappa_1 - \delta_p] (1 - e^{-K_1T})} \right] \quad (2.6)
\]
for all \( T > 0, g \in B^+_E(\mathbb{R}^d) \) and \( x, y \in \mathbb{R}^d \). Now we shall apply the relation \((2.4)\) to show that the semigroup \( P_t \) associated with the SDE \((1.1)\) also satisfies the same Harnack inequalities (possibly with different constants).
(1) Take $T > 0$ and $x, y \in \mathbb{R}^d$. For $f \in \mathcal{B}_b^+(\mathbb{R}^d)$ with $f \geq 1$, set $g = f \circ \Psi^{-1}$. We have by (2.4) and (2.5),

$$P_T \log f(y) = \hat{P}_T \log g(\Psi(y)) \leq \log \hat{P}_T g(\Psi(x)) + \frac{K_1|\Psi(x) - \Psi(y)|^2}{2\kappa_1^2(1 - e^{-K_1T})}.$$ 

Since $|\Psi(x) - \Psi(y)| \leq |x - y| + |\psi(x) - \psi(y)| \leq 3|x - y|/2$, again by (2.4), we have

$$P_T \log f(y) \leq P_T f(x) + \frac{2K_1|x - y|^2}{\kappa_1^2(1 - e^{-K_1T})}, \quad f \in \mathcal{B}_b^+(\mathbb{R}^d) \text{ with } f \geq 1.$$ 

(2) In the same way, for $f \in \mathcal{B}_b^+(\mathbb{R}^d)$, let $g = f \circ \Psi^{-1}$. Then

$$(P_Tf(y))^p = (\hat{P}_Tg(\Psi(y))) \leq [\hat{P}_Tg^p(\Psi(x))] \exp \left[ \frac{K_1\sqrt{p}(\sqrt{p} - 1)|\Psi(x) - \Psi(y)|^2}{4\delta_p((\sqrt{p} - 1)\kappa_1 - \delta_p)(1 - e^{-K_1T})} \right]$$

$$\leq [P_Tf^p(x)] \exp \left[ \frac{K_1\sqrt{p}(\sqrt{p} - 1)|x - y|^2}{\delta_p((\sqrt{p} - 1)\kappa_1 - \delta_p)(1 - e^{-K_1T})} \right].$$

The proof is complete. \hfill \Box

3 Log-Harnack inequality for SDE with time-dependent singular drift

In this section we allow the coefficients to depend on time. More precisely, let $\sigma : [0, T] \times \mathbb{R}^d \to \mathbb{R}^m \otimes \mathbb{R}^d$ and $b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ be measurable functions, and consider the Itô SDE

$$dX_t = \sigma_t(X_t) \, dW_t + b_t(X_t) \, dt, \quad X_0 = x \in \mathbb{R}^d. \tag{3.1}$$

Assume that the drift coefficient $b$ satisfies

$$\frac{d}{p} + \frac{2}{q} < 1. \tag{3.2}$$

for some $p > d$ and $q > 2$. The SDE (3.1) with diffusion coefficient $\sigma = Id$ (hence $m = d$) and drift coefficient $b$ satisfying (3.2) was first studied by Krylov and Röckner [6], where the existence of unique strong solution was proved. X. Zhang [17] extended their result to the more general SDE (3.1) with variable diffusion coefficient $\sigma$. Furthermore, it was shown in [18] (see also [3]) that the SDE (3.1) generates a unique stochastic flow $X_t$ of homeomorphisms on $\mathbb{R}^d$, provided that the diffusion coefficient $\sigma$ satisfies the following conditions:

$(H_1^\sigma)$ $\sigma_t(x)$ is uniformly continuous in $x \in \mathbb{R}^d$ locally uniformly with respect to $t \in \mathbb{R}_+$, and there exist positive constants $K$ and $\delta$ such that for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\delta|y|^2 \leq |\sigma_t^*(x) y|^2 \leq K|y|^2,$$

where $\sigma_t^*(x)$ is the transposition;

$(H_2^\sigma)$ $|\nabla \sigma_t| \in L^q_{\text{loc}}(\mathbb{R}^d; L^p(\mathbb{R}^d))$ with the same $p, q$ as in (3.2), where $\nabla$ denotes the generalized gradient with respect to $x$. 

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Our purpose in this section is to show that the associated semigroup
\[ P_t f(x) = \mathbb{E} f(X_t(x)), \quad f \in C^2_b(\mathbb{R}^d), \quad t \geq 0 \] (3.3)
satisfies the log-Harnack inequality. Recall that J. Shao [9] studied the Harnack inequalities for
\( P_t \) generated by (3.1), but there are some additional constants in the inequalities [9, (2.3) and
(2.4)]. In this paper, we shall improve the log-Harnack inequality [9, (2.3)] by removing the
extra constant. Our method is based on the \( L^2 \)-gradient estimate on the semigroup \( P_t \). For the
moment, we are unable to improve the Harnack inequality [9, (2.4)] with powers since we do not
have the \( L^1 \)-gradient estimate on the semigroup, \( \). 

We shall first prove the log-Harnack inequality for the semigroup associated to the following
\( \text{Itō SDE without drift:} \)
\[ dY_t = \sigma_t(Y_t) \, dW_t, \quad Y_0 = x, \] (3.4)
where \( \sigma \) verifies \( (H^\sigma_1) \) and \( (H^\sigma_2) \). In the sequel we denote by \( T_{s,t} \) the two-parameter semigroup
associated to (3.4) defined by
\[ T_{s,t} f(x) = \mathbb{E}(f(Y_t)|Y_s = x). \]
For \( t \geq 0 \), define the time-dependent second order differential operator
\[ L_t f(x) = \frac{1}{2} \text{Tr} [\sigma_t(x)\sigma^*_t(x)\nabla^2 f(x)] + \langle b_t(x), \nabla f(x) \rangle, \quad f \in C^2_b(\mathbb{R}^d), \]
where \( \nabla^2 f \) is the Hessian matrix of \( f \). Then we have the well-known Kolmogorov equations:
\[ \partial_s T_{s,t} f = -L_s T_{s,t} f, \quad \partial_t T_{s,t} f = T_{s,t} L_t f. \]

Define
\[ \Gamma(t)(f, g) = \frac{1}{2} \{ L_t(fg) - gL_t f - fL_t g \}, \]
\[ \Gamma_2(t)(f, g) = \frac{1}{2} \{ L_t \Gamma(t)(f, g) - \Gamma(t)(fg, L_t f) - \Gamma(t)(f, L_t g) \}. \]

Then
\[ \Gamma(t)(f, g) = \frac{1}{2} \langle \sigma_t(x) \nabla f, \sigma^*_t(x) \nabla g \rangle. \]

Let \( \{ \rho_n \}_{n \geq 1} \) be a family of mollifier on \( \mathbb{R}^d \) and set \( \sigma_t^n(x) = \sigma_t * \rho_n(x), \quad n \geq 1 \). We consider
the smooth equation
\[ dY^n_t = \sigma_t^n(Y^n_t) \, dW_t, \quad Y^n_0 = x. \] (3.5)
Following the arguments in the proof of [18, Theorem 3.5], we can show that
\[ C_1 := \sup_{n \geq 1} \sup_{t \leq T} \mathbb{E}( |\nabla Y^n_t(x)|^2 ) < +\infty. \] (3.6)
Then for any \( f \in C^1_b(\mathbb{R}^d) \) and \( x, y \in \mathbb{R}^d \), by the mean value formula,
\[ f(Y^n_t(x)) - f(Y^n_t(y)) = \int_0^1 \langle \nabla f(Y^n_t(y + r(x - y))), \nabla Y^n_t(y + r(x - y)) \rangle \, dr. \]
Therefore
\[ |f(Y^n_t(x)) - f(Y^n_t(y))| \leq |x - y| \int_0^1 |\nabla f(Y^n_t(y + r(x - y)))| \cdot |\nabla Y^n_t(y + r(x - y))| \, dr. \]
Cauchy’s inequality and (3.6) implies that

\[ |\mathbb{E}(Y_t^n(x)) - \mathbb{E}(Y_t^n(y))| \leq \sqrt{C_1} |x - y| \int_0^1 \left( \mathbb{E}|\nabla f(Y_t^n(y + r(x - y))| \right)^2 dr. \]

Moreover, by [18, (3.7)] we have \( \lim_{n \to \infty} \mathbb{E}|Y_t^n(x) - Y_t(x)| = 0 \). Thus by dominated convergence theorem, letting \( n \) tend to \( \infty \) in the above inequality yields

\[ |\mathbb{E}(Y_t(x)) - \mathbb{E}(Y_t(y))| \leq \sqrt{C_1} |x - y| \int_0^1 \left( \mathbb{E}|\nabla f(Y_t(y + r(x - y))| \right)^2 dr. \]

Now we let \( y \to x \) and obtain

\[ |\nabla T_t f(x)|^2 \leq C_1 T_t |\nabla f|^2(x), \quad x \in \mathbb{R}^d. \]

Similarly, we have for all \( s \leq t \),

\[ |\nabla T_{s,t} f(x)|^2 \leq C_1 T_{s,t} |\nabla f|^2(x), \quad x \in \mathbb{R}^d. \tag{3.7} \]

Now standard arguments leads to the log-Harnack inequality for the semigroup \( T_{s,t} \).

**Proposition 3.1.** Assume that \( \sigma \) verifies \((H^*_1)\) and \((H^*_2)\). Then for any \( f \in B_0^+(\mathbb{R}^d) \) with \( f \geq 1 \), we have

\[ T_{s,t} \log f(y) \leq \log T_{s,t} f(x) + \frac{C_1|y - x|^2}{2\delta(t - s)}, \quad \text{for all } x, y \in \mathbb{R}^d, s \leq t, \tag{3.8} \]

where \( \delta \) is the constant in \((H^*_1)\).

**Proof.** Take \( f \geq 1 \). For \( x, y \in \mathbb{R}^d \) and \( s \leq u \leq t \), let \( \gamma_u = (y - x) \frac{u - s}{t - s} + x \). Applying Itô’s formula, we have

\[
\begin{align*}
\text{d} \log T_{u,t} f(Y_u) = & \langle \nabla \log T_{u,t} f(Y_u), \sigma_u(Y_u) \text{d}B_u \rangle + L_u \log T_{u,t} f(Y_u) \text{d}u - \frac{L_u T_{u,t} f(Y_u)}{T_{u,t} f(Y_u)} \text{d}u \\
= & \langle \nabla \log T_{u,t} f(Y_u), \sigma_u(Y_u) \text{d}B_u \rangle - \Gamma(u)(T_{u,t} f)(Y_u) \text{d}u,
\end{align*}
\]

where the last equality follows by

\[ L_u \log T_{u,t} f(\gamma_u) = \frac{L_u T_{u,t} f(Y_u)}{T_{u,t} f} - \frac{\Gamma(u)(T_{u,t} f)}{(T_{u,t} f)^2}. \]

Let

\[ \tau_n = \inf\{t \geq 0 : |Y_t| \geq n\}, \quad n \geq 1. \]

Then, by integrating w.r.t. \( u \), we get

\[ \mathbb{E} \log T_{u \wedge \tau_n, t} f(Y_{u \wedge \tau_n}) - \log T_{s,t} f(x) = -\int_0^{u \wedge \tau_n} \Gamma(u)(T_{u,t} f)(Y_u) \text{d}u. \tag{3.9} \]

It is known that the process is non-explosive; hence \( \tau_n \to \infty \) as \( n \to \infty \). Thus, passing to the limit as \( n \to \infty \), and applying the dominated convergence theorem to the left hand side of (3.9) and the monotone convergence theorem to the right hand side of (3.9), we have

\[ T_{s,u} \log T_{u,t} f(x) - \log T_{s,t} f(x) = -\int_0^u \Gamma(u)(T_{u,t} f)(Y_u) \text{d}u, \quad u \in [s, t]. \tag{3.10} \]
Now for \( x, y \in \mathbb{R}^d \), let \( \gamma_u = (y - x) \frac{u - s}{t - s} + x \). The identity (3.10) implies that \( [s, t] \ni u \mapsto T_{s,u} \log T_{u,t}f(\gamma_u) \) is absolutely continuous, thus by \( (H_2^p) \), we have
\[
\frac{d}{du} T_{s,u} \log T_{u,t}f(\gamma_u) \leq T_{s,u} L_u \log T_{u,t}f(\gamma_u) = T_{s,u} \left( \frac{L_u T_{u,t}f(\gamma_u)}{T_{u,t}f(\gamma_u)} \right) + |\nabla T_{s,u} \log T_{u,t}f(\gamma_u)\gamma_u| \\
\leq -T_{s,u} \left( \frac{1}{(T_{u,t}f)^2} \right) (\gamma_u) + |\nabla T_{s,u} \log T_{u,t}f(\gamma_u)\gamma_u| \\
\leq -\frac{\delta}{2} T_{s,u} |\nabla T_{u,t}f|^2(\gamma_u) + |\gamma_u| (C_1 T_{s,u} |\nabla T_{u,t}f|^2(\gamma_u))^\frac{1}{2} \\
\leq C_1^2 |\gamma_u|^2, \quad \mathcal{L}^1\text{-a.e.,}
\]
where in the third inequality we have used (3.7). Integrating from \( s \) to \( t \) gives us the log-Harnack inequality (3.8). \( \square \)

It remains to transfer the above result to the general Itô SDE (3.1) with drift. Before moving on, we introduce two function spaces: for \( p, q \geq 1 \) and \( S, T > 0 \), let
\[
L^p_S(S,T) = L^p([S,T], \mathbb{L}^p(d\mathbb{R}^d)) \quad \text{and} \quad \mathbb{H}^{2,q}_p(S,T) = L^q([S,T], H^2_p(d\mathbb{R}^d)),
\]
where \( H^2_p(d\mathbb{R}^d) \) is the usual Sobolev space.

We shall need the following preparations which are taken from [18, pp.1110–1111]. Assume that \( \sigma \) satisfies \( (H^p) \) and \( b \in L^q(\mathbb{R}^d, \mathbb{L}^p(d\mathbb{R}^d)) \) with \( p, q \) verifying (3.2). Fix \( T_0 > 0 \). For any \( 0 \leq S \leq T \) with \( T - S \leq T_0 \), let \( u(t, x) \) be the solution to the backward parabolic equation
\[
\partial_t u^i(t, x) + L_t u^i(t, x) + b^i(t, x) = 0, \quad u^i(T, x) = 0, \quad 1 \leq i \leq d.
\] (3.11)

Then by [18, Theorem 5.1] (see also [6, Theorem 10.3 and Remark 10.4]), one has
\[
C_2 := \sup_{S \in [0, (T-T_0)^+]} \left( \|\partial_t u\|_{L^p_S(S,T)} + \|u\|_{\mathbb{H}^{2,q}_p(S,T)} \right) < +\infty. \quad (3.12)
\]

It follows from [6, Lemma 10.2] that the function \( (t, x) \mapsto \nabla u(t, x) \) is Hölder continuous and for fixed \( \delta \in \left( 0, \frac{1}{2} - \frac{d}{2p} - \frac{1}{q} \right) \), there exists \( C_3 > 0 \) depending on \( p, q, \delta, T_0 \) such that
\[
\sup_{(t, x) \in [S, T] \times \mathbb{R}^d} |\nabla u(t, x)| \leq C_3 T_0^\delta. \quad (3.13)
\]

Define \( \Phi_t(x) = x + u(t, x), \ (t, x) \in [S, T] \times \mathbb{R}^d \). It is easy to see that
\[
\partial_t \Phi_t(x) + L_t \Phi_t(x) = 0, \quad \Phi_T(x) = x.
\]

Moreover, if \( T_0 \) is small enough, we deduce from (3.13) that for all \( t \in [S, T] \),
\[
\frac{1}{2} |x - y| \leq |\Phi_t(x) - \Phi_t(y)| \leq \frac{3}{2} |x - y|, \quad \text{for all} \ x, y \in \mathbb{R}^d. \quad (3.14)
\]

Therefore \( \Phi_t \) is a diffeomorphism on \( \mathbb{R}^d \). The following result is proved in [18, Lemma 4.3].

**Lemma 3.2** \( (Zvonkin-type transformation) \). Let \( X_t \) be a \( \mathbb{R}^d \)-valued \( \mathcal{F}_t \)-adapted continuous process satisfying
\[
\mathbb{P} \left\{ \omega : \int_S^T \left( |b_s(X_s(\omega))| + |\sigma_s(X_s(\omega))| \right)^2 \, ds < +\infty \right\} = 1.
\]

Then \( X_t \) solves the equation (3.1) on the time interval \( [S, T] \) if and only if \( Y_t = \Phi_t(X_t) \) solves the following SDE on \( [S, T] \):
\[
dY_t = \Sigma_t(Y_t) \, dW_t, \quad (3.15)
\]
where \( \Sigma_t^k(y) = (\partial_t \Phi_t \cdot \sigma_{-1}(y) \circ \Phi_t^{-1}(y), \ 1 \leq i \leq d, 1 \leq k \leq m. \)
With Proposition 3.1 and Lemma 3.2 in hand, we can now prove the main result of this section.

**Theorem 3.3 (Log-Harnack inequality).** Assume that the diffusion coefficient \( \sigma \) fulfills \((H_1^\sigma)\) and \((H_2^\sigma)\), and \(b \in L^p(\mathbb{R}^d, L^q(\mathbb{R}^d)) \) with \( p, q \) verifying \((3.2)\). Then the semigroup \( P_{s,t} \) associated to \((3.1)\) satisfies the log-Harnack inequality below:

\[
P_{s,t} \log f(y) \leq \log P_{s,t} f(x) + \frac{\hat{C}_1 |y - x|^2}{\delta(t - s)}, \tag{3.16}\]

where \(\hat{C}\) is some positive constant and \(\delta\) is the parameter appearing in \((H_1^\sigma)\).

**Proof.** We first check that the matrix valued function \( \Sigma \) given in Lemma 3.2 satisfies \((3.1)\), \((3.2)\), \((3.3)\) and \((3.4)\), \((3.5)\) with \(\xi_1 > 0\) and \(\xi_2 > 0\) such that for all \( (t, x) \in [0, T] \times \mathbb{R}^d \), it holds

\[
\frac{1}{2} \leq \| \nabla \Phi_t(x) \| \leq \frac{3}{2}, \quad \text{for all} \quad (t, x) \in [S, T] \times \mathbb{R}^d.
\]

Since \(\Sigma_t = (\nabla \Phi_t \sigma_t) \circ \Phi_t^{-1}\), we deduce from \((H_1^\sigma)\) that

\[
\frac{1}{4} \delta |y|^2 \leq |\Sigma_t^*(x)y|^2 \leq \frac{9}{4} K |y|^2, \quad \text{for all} \quad (t, x) \in [0, T] \times \mathbb{R}^d \quad \text{and} \quad y \in \mathbb{R}^d.
\]

Thus \((H_1^\sigma)\) holds with new constants \(\frac{1}{4} \delta\) and \(\frac{9}{4} K\). For the second condition \((H_2^\sigma)\), we note that

\[
\partial_t \Sigma_t^{ik}(y) = [(\partial_i \partial_j \Phi_t^* \sigma_t^{jk} + \partial_j \Phi_t^* \cdot \partial_i \sigma_t^{jk}) \circ \Phi_t^{-1}(y)] \cdot \partial_t \Phi_t^{l,-1}(y).
\]

By \((3.12)\), \((3.14)\) and \((H_2^\sigma)\), we conclude that \(\| \partial_t \Sigma_t^{ik} \|_{L^2(K)} < +\infty\). That is, \((H_2^\sigma)\) also holds but on a finite interval \([0, T_0]\).

Denote by \(\tilde{T}_{s,t}\) the semigroup associated to the new SDE \((3.15)\) without drift. We can apply Proposition 3.1 to obtain that, for any \( f \in B^+_b(\mathbb{R}^d) \) with \( f \geq 1 \) and any \( 0 \leq s \leq t \) with \( t - s \leq T_0 \), it holds

\[
\tilde{T}_{s,t} \log f(y) \leq \log \tilde{T}_{s,t} f(x) + \frac{\tilde{C}_1 |y - x|^2}{\delta(t - s)}, \quad \text{for all} \quad x, y \in \mathbb{R}^d,
\]

where \(\hat{C}_1 > 0\) is some constant. Now, by \((3.14)\) and Lemma 3.2, we can follow the idea in the proof of Theorem 1.1 to transfer the above log-Harnack inequality \((3.17)\) to the one with the semigroup \( P_{s,t} \) associated to \((3.1)\). Thus there exists a positive constant \(\tilde{C}_1 > 0\), such that for any \( f \in B^+_b(\mathbb{R}^d) \) with \( f \geq 1 \) and any \( 0 \leq s \leq t \) with \( t - s \leq T_0 \), it holds

\[
P_{s,t} \log f(y) \leq \log P_{s,t} f(x) + \frac{\tilde{C}_1 |y - x|^2}{\delta(t - s)}, \quad \text{for all} \quad x, y \in \mathbb{R}^d.
\]

If \( t - s \in (T_0, 2T_0) \), then by the semigroup property and Jensen’s inequality, we have

\[
P_{s,t} \log f(y) = P_{s+T_0,t}(P_{s,s+T_0} \log f)(y) \leq P_{s+T_0,t} [\log (P_{s,s+T_0} f)](y)
\]

\[
\leq \log [P_{s+T_0,t}(P_{s,s+T_0} f)](x) + \frac{\tilde{C}_1 |y - x|^2}{\delta(t - s)}
\]

\[
= \log (P_{s,t} f)(x) + \frac{\tilde{C}_1 |y - x|^2}{\delta(t - s)},
\]

where in the second inequality we have used \((3.17)\). Repeating this procedure, we conclude that the log-Harnack inequality \((3.17)\) holds for all \( 0 \leq s \leq t \).

\[\square\]
4 Harnack Inequalities for SDEs Driven by Symmetric Stable Processes

In this section, we consider the following SDE
\[ dX_t = b(X_t) \, dt + dZ_t, \quad (4.1) \]
where \( Z_t \) is a symmetric \( \alpha \)-stable process with \( \alpha \geq 1 \) and \( b \in C^\beta_b(\mathbb{R}^d; \mathbb{R}^d) \) with \( \beta > 1 - \frac{\alpha}{2} \). In [7, Theorem 1.1], Priola proved that the solution \( X_t \) to (4.1) is still a flow of homeomorphisms which are \( C^1 \)-functions on \( \mathbb{R}^d \). His proof, similar to that in [5], is based on the so-called Itô–Tanaka trick which transforms the SDE into a new one with regular drift coefficient; see [7, (4.10) or (4.14)]. We tried to establish the Harnack inequalities for the semigroup \((P_t)_{t \geq 0}\) associated to SDE (4.1) in this framework, but in vain. The difficulty comes from the term on the right hand side of [7, (4.14)], which involves an integral with respect to the Poisson random measure.

A possible approach to bypass the difficulty is to use the regularization approximations of the underlying subordinator; see [14] for the recent study of dimension-free Harnack inequalities for a class of stochastic equations driven by a Lévy noise containing a subordinator Brownian motion. In this case, since the vector field \( b \) is time-independent, the drift coefficient in the new SDE is variable-separated, and the diffusion coefficient is the identity matrix (cf. [19, (2.3)] or [14, (3.2)]). At first glance, it seems that one can establish the Harnack inequalities in the same way as above by using the transform in [4, page 14]. However, this requires an explicit expression for the constant \( C \) in the Schauder estimate (see [4, Theorem 2]), especially its dependence on the time-change factor. After checking the proofs of [4, Theorem 2 and Lemma 4], unfortunately, we did not obtain useful estimates on \( C \).

On the other hand, by using the explicit heat kernel estimates, we can prove

**Proposition 4.1.** There exists a constant \( C > 1 \) such that for any \( f \in B^b_+(\mathbb{R}^d) \), \( T > 0 \) and \( x, y \in \mathbb{R}^d \),
\[ P_T f(x) \leq C \left( 1 + \frac{|x - y|}{(T \wedge 1)^{1/\alpha}} \right)^{d+\alpha} P_T f(y). \quad (4.2) \]

**Proof.** Note that the drift term \( b \) is bounded, and so it belongs to the Kato class (see e.g. [2, Definition 1.1]). Then, for \( \alpha > 1 \), according to [2, Corollary 1.3], the process \( X_t \) has a jointly continuous transition density function \( p(t, x, y) \) with respect to the Lebesgue measure on \( \mathbb{R}^d \). Moreover, there exists a constant \( c \geq 1 \) such that for all \( (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \),
\[ c^{-1} \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right) \leq p(t, x, y) \leq c \left( t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}} \right). \quad (4.3) \]

On the other hand, noticing that \( b \in C^\beta_b(\mathbb{R}^d; \mathbb{R}^d) \) with \( \beta > 1 - \frac{\alpha}{2} \), we know from [16, Theorem 1.1] (see also [16, Theorem 3.6]) that (4.3) also holds for \( \alpha = 1 \).

Furthermore, having (4.3) at hand and following the proof of [15, Lemma 2.1], we can get that for any \( t \in (0, 1] \) and \( x, y, z \in \mathbb{R}^d \),
\[ \frac{p(t, x, z)}{p(t, y, z)} \leq 2^{\alpha+d} c^2 \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha}. \quad (4.4) \]
Therefore, for any $f \in B^+_b(\mathbb{R}^d)$, $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,

$$P_t f(x) = \int f(z) p(t, x, z) \, dz = \int f(z) \frac{p(t, x, z)}{p(t, y, z)} p(t, y, z) \, dz$$

$$\leq \left( \max_{z \in \mathbb{R}^d} \frac{p(t, x, z)}{p(t, y, z)} \right) \int f(z) p(t, y, z) \, dz$$

$$= C \left( 1 + \frac{|x - y|}{t^{1/\alpha}} \right)^{d+\alpha} P_t f(y),$$

where $C$ is a positive constant independent of $x, y, z$ and $t$. For any $t > 0$, we write

$$P_t f = P_{t \wedge 1} P_{(t-1)\wedge 1} f.$$

This, together with the inequality above, yields the required assertion. 

Surely, the Harnack inequality (4.2) is not satisfactory in the sense that $C > 1$, which means that such inequality is not sharp for the case $x = y$. Nonetheless, since the process has the transition density function, it has the strong Feller property, and so even for $C > 1$, we still have some applications, e.g., long time behaviors and properties of invariant measure.

**References**


[18] Xicheng Zhang, Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients. 