Track-Following Control of Four-Bar Structured HDD via Parameter-Dependent Low-Frequency Precompensation

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This paper considers track-following control of four-bar structured hard disk drive (HDD) systems. Compared with the traditional single-arm HDD, the four-bar structure potentially leads to higher achievable closed-loop bandwidth, but also brings challenges in control design due to nonlinearity. In track-following control design, exact linearization is impossible in practice due to mechanic resonances and measurement constraints. In this paper, instead of exact linearization, we propose a kind of prefilter to compensate for the low-frequency nonlinearities so that the compensated system has the same linear approximation at different track positions subjected to different high-frequency resonances. With this prefilter, many existing track-following control methods can be applied to the compensated plant. In this paper, $H_{\infty}$ loop shaping is used and simulation results show that the resulting controller preserves similar track-following performance at different track positions.

Index Terms—Four-bar structure, hard disk drive, precompensation, track-following control.

I. INTRODUCTION

Mechanic resonances are one of the main factors limiting the performance of hard disk drive servo systems. For instance, the achievable closed-loop bandwidth is limited by the first resonance frequency due to interpolation constraints [1]. In practice, the performance is mainly affected by the first or the second torsion modes and sway modes which are related to the stiffness of the suspension [2]. On one hand, stiffer suspensions are preferred in order to push the closed-loop bandwidth higher. On the other hand, suspensions are expected to be light in order to achieve fast response in the track-seeking stage. Drive designers have to consider the tradeoff between the mass and the stiffness of the suspension. One solution to achieve a better tradeoff is to use lighter but stiffer materials. Another solution is to adopt new mechanical structures.

Fig. 1 is a sketch map for the four-bar structured hard disk drive considered in this paper whose design is currently patent pending. Different from the traditional single-arm structure, the four-bar structured HDD suspension is driven by the actuator through a connection link, as shown in Fig. 1. The main benefit of this structure is that it provides additional stiffness without degrading the moment requirement and thus achieves better tradeoff between mass and stiffness.

Though the four-bar structure potentially leads to a higher achievable closed-loop bandwidth, it also brings nonlinearities which greatly challenge the control design. Unlike the traditional single-arm structured suspension whose dynamical model can be well approximated by a single linear system within the whole span of the R/W head, the dynamics of a four-bar structured model at different track positions differ greatly and thus are impossible to be captured by a single linear model. In the track-seeking stage where low-frequency dynamics play the leading role, exact linearization based on rigid model is a possible solution to obtain a seeking controller which preserves track-seeking performance at different seeking scenarios. However, in the track-following stage when high-frequency dynamics becomes significant, exact linearization is unrealistic due to mechanic resonances and measurement constraints.

Instead of exact linearization, we propose a parameter-dependent prefilter in this paper to compensate for the low-frequency nonlinearities. After compensation, the compensated system has approximately the same rigid model at different track positions subjected to different high-frequency resonances. The main advantage of this prefilter is that it only depends on the target track position and thus very easy in implementation. With this prefilter, many existing track-following design methods can be applied to the compensated system. For existing track-following design methods, we refer to [3]–[10], etc. In this paper, $H_{\infty}$ loop shaping method is applied to show the validity of the prefilter.

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Simulation results show that, together with the prefilter, the $H_\infty$ controller applies to almost all the track following situations.

This paper is arranged as follows. In Section II, the dynamic equations of the four-bar system with flexible and rigid joints are introduced respectively. Section III proposes the detailed design process, especially the design of the prefilter. Section IV gives the simulation results and some concluding remarks are made in Section V.

II. DYNAMIC EQUATIONS

The planar four-bar system considered in this paper is depicted in Fig. 2. It consists of the driving bar $A_1A_2$, the driven bar $A_1A_2$, the connection bar $A_2A_3$ and the forth bar $A_1A_4$ which is fixed with the pedestal. The joint $A_4$ is the driving joint where a control torque $\tau$ is provided by the actuator and the R/W head is fixed somewhere on the driven bar $A_1A_2$. The joint $A_1$ is assumed to be flexible and the flexibility is characterized by springs with elastic coefficients $k_x$ and $k_y$, respectively. The corresponding rigid structure is depicted in Fig. 3.

A. Model With Flexible Joint

Note that this structure is of 3-DOF (degree of freedom), we choose $\theta_i, i = 1, 2, 4$, as the generalized coordinates. Define $\theta = [\theta_1, \theta_2, \theta_4]^T$, $b = [0, 0, 1]^T$, then the dynamics of this system can be established as

$$M(\theta)\ddot{\theta} + C(\theta, \dot{\theta})\dot{\theta} + \frac{\partial V(\theta)}{\partial \theta} + \frac{\partial F(\dot{\theta})}{\partial \dot{\theta}} = b\tau \tag{1}$$

where the positive-definite inertia matrix $M(\theta)$ is given by

$$M(\theta) = [M_{ij}(\theta)]_{3 \times 3}$$

with

$$M_{11} = 2(J_{1df} + J_{1du} \cos \theta_2)$$
$$M_{12} = J_{1cf} + J_{1ce} \cos \theta_2$$
$$M_{13} = J_{2ef} \cos(\theta_1 + \theta_2) + J_{2ev} \cos(\theta_1 + \theta_2 + \theta_4)$$
$$M_{22} = 2J_{2df}$$
$$M_{23} = J_{3cv} \cos(\theta_1 + \theta_2 + \theta_4)$$
$$M_{33} = 2J_{3df},$$

while $J_{1df}, J_{1cf}, J_{1ce}, J_{2df}, J_{2cf}, J_{2ev}, J_{2ef}, J_{3cv}, J_{3cf}$ and $J_{3ce}$ are given by

$$J_{1df} = \frac{1}{2} \left( m_2l_2^2 + J_{1c} + m_1l_1^2 + m_1l_2^2 + J_{1c} \right)$$
$$J_{1du} = -m_1l_1l_2$$
$$J_{2df} = \frac{1}{2} \left( m_2l_2^2 + J_{2c} + m_1l_1^2 \right)$$
$$J_{2cf} = \frac{1}{2} \left( m_2l_2^2 + J_{2c} + m_1l_1^2 \right)$$
$$J_{2ev} = -m_1l_1l_2$$
$$J_{2ef} = m_2l_2^2 + J_{2c} + m_1l_1^2$$
$$J_{3cv} = m_1l_1l_2$$

and the other parameters are listed in Table I. The matrix

$$C(\theta, \dot{\theta}) = [c_{ik}]_{3 \times 3}$$

is uniquely determined by $M(\theta)$ through

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_1$ (m)</td>
<td>length of the first bar $A_1A_2$</td>
<td>$1.8866 \times 10^2$</td>
</tr>
<tr>
<td>$l_2$ (m)</td>
<td>length of the second bar $A_2A_3$</td>
<td>$9.835 \times 10^3$</td>
</tr>
<tr>
<td>$l_3$ (m)</td>
<td>length of the third bar $A_1A_4$</td>
<td>$1.4338 \times 10^2$</td>
</tr>
<tr>
<td>$c$ (m)</td>
<td>distance between $A_1$ and the</td>
<td>$3.7714 \times 10^2$</td>
</tr>
<tr>
<td></td>
<td>equilibrium position $O$ of $A_i$</td>
<td></td>
</tr>
<tr>
<td>$l_{c1}$ (m)</td>
<td>distance from $A_2$ to the</td>
<td>$1.41 \times 10^2$</td>
</tr>
<tr>
<td></td>
<td>center of mass of $A_1A_2$</td>
<td></td>
</tr>
<tr>
<td>$l_{c2}$ (m)</td>
<td>distance from $A_1$ to the</td>
<td>$4.918 \times 10^3$</td>
</tr>
<tr>
<td></td>
<td>center of mass of $A_2A_3$</td>
<td></td>
</tr>
<tr>
<td>$l_{c3}$ (m)</td>
<td>distance from $A_4$ to the</td>
<td>$1.406 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>center of mass of $A_1A_4$</td>
<td></td>
</tr>
<tr>
<td>$m_1$ (kg)</td>
<td>mass of the first bar $A_1A_2$</td>
<td>$2.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>$m_2$ (kg)</td>
<td>mass of the second bar $A_2A_3$</td>
<td>$1.1 \times 10^{-3}$</td>
</tr>
<tr>
<td>$m_3$ (kg)</td>
<td>mass of the third bar $A_1A_4$</td>
<td>$1.78 \times 10^{-2}$</td>
</tr>
<tr>
<td>$J_{ce}$ (kg·m$^2$)</td>
<td>inertia of moment of $A_1A_2$</td>
<td>$1.671 \times 10^{-7}$</td>
</tr>
<tr>
<td></td>
<td>w.r.t. mass center</td>
<td></td>
</tr>
<tr>
<td>$J_{cf}$ (kg·m$^2$)</td>
<td>inertia of moment of $A_1A_2$</td>
<td>$1.948 \times 10^{-9}$</td>
</tr>
<tr>
<td></td>
<td>w.r.t. mass center</td>
<td></td>
</tr>
<tr>
<td>$J_{cv}$ (kg·m$^2$)</td>
<td>inertia of moment of $A_1A_4$</td>
<td>$5.992 \times 10^{-8}$</td>
</tr>
<tr>
<td></td>
<td>w.r.t. mass center</td>
<td></td>
</tr>
<tr>
<td>$k_x, k_y$ (N·m)</td>
<td>elastic coefficient of the springs</td>
<td>$9.78 \times 10^6$</td>
</tr>
<tr>
<td>$\sigma_i$</td>
<td>Rayleigh dissipation coefficient of joint $A_i$</td>
<td>$3 \times 10^5$</td>
</tr>
</tbody>
</table>
\[ v_{ik} = \frac{1}{2} \sum_{j=1}^{3} \left( \frac{\partial_j M_{ik}}{\partial b_k} + \frac{\partial_k M_{ik}}{\partial b_j} - \frac{\partial_k M_{ij}}{\partial b_i} \right)[\hat{\theta}]_j \]

where \( \frac{\partial_j}{\partial b_k} \) means partial derivative with respect to the \( j \)th component of \( b \) and \( (\hat{\theta})_j \) represents the \( j \)th component of \( \hat{\theta} \). \( V(\theta) \) is the potential energy given by

\[
V(\theta) = -\frac{1}{2} k_x x^2 + \frac{1}{2} k_y y^2 - \frac{1}{2} k_x c_0 - l_1 \cos \theta_1 - l_2 \cos(\theta_1 + \theta_2) - l_3 \cos \theta_4 \]
\[
+ \frac{1}{2} k_y l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) - l_3 \sin \theta_4 \]

where \( x \) and \( y \) are, respectively, the horizontal and vertical displacement of joint \( A_1 \) from the equilibrium position \( O, c_0 \) is the distance between the joint \( A_4 \) and the equilibrium position \( O \) of joint \( A_1 \), \( k_x \) and \( k_y \) are elastic coefficients. The Rayleigh dissipation function \( F(\dot{\theta}) \) is of the form

\[
F(\dot{\theta}) = \frac{1}{2} \left( \sigma_1 \dot{\theta}_1^2 + \sigma_2 \dot{\theta}_2^2 + \sigma_3 \dot{\theta}_3^2 + \sigma_4 \dot{\theta}_4^2 \right)
\]
\[
= \frac{1}{2} \dot{\theta}^T K \dot{\theta}
\]

with \( \sigma_i \) positive Rayleigh dissipation coefficient of joint \( A_i \) and

\[
K = \begin{bmatrix}
\sigma_1 + \sigma_3 & \sigma_3 & \sigma_3 \\
* & \sigma_2 + \sigma_3 & \sigma_3 \\
* & * & \sigma_4 + \sigma_3
\end{bmatrix}.
\]

**B. Rigid Model**

In rigid case, the DOF is one and \( \theta_4 \) is chosen as the generalized coordinate. See Fig. 3. The dynamic equation is established as

\[
D(\theta) \ddot{\theta}_4 + \frac{1}{2} \hat{D}(\theta) \dot{\theta}_4 + \frac{\partial F(\dot{\theta})}{\partial \theta_4} = \tau
\]

where

\[
D(\theta) = J_{3c} + m_3 \dot{l}_{3c}^2 + m_3 l_3^2
\]
\[
+ \left( m_1 (l_1 - l_{1c}) \right) \frac{l_1^2 \sin^2 \theta_2}{l_1 \sin \theta_2}
\]
\[
+ \left( m_2 \dot{l}_{2c} + J_{2c} \right) \frac{l_2^2 \sin^2 (\theta_1 + \theta_4)}{l_2 \sin \theta_2}
\]
\[
+ \frac{2 m_2 l_2 \cos (\theta_1 + \theta_2 + \theta_4) \sin (\theta_1 + \theta_4)}{l_2 \sin \theta_2}
\]

and

\[
F(\dot{\theta}) = \frac{1}{2} \dot{\theta}^T K \dot{\theta} = \frac{1}{2} \hat{K}(\dot{\theta}) \dot{\theta}_4^2
\]

with \( \hat{K}(\dot{\theta}) \) defined in (2). Note that in the rigid case \( \theta \) is subjected to the geometric constraints

\[
\lambda_1(\theta) = 0, \quad \lambda_2(\theta) = 0
\]

where

\[
\begin{align*}
\lambda_1(\theta) & := -l_1 \cos \theta_1 - l_2 \cos(\theta_1 + \theta_2) - l_3 \cos \theta_4 - c_0 \\
\lambda_2(\theta) & := l_1 \sin \theta_1 - l_2 \sin(\theta_1 + \theta_2) - l_3 \sin \theta_4.
\end{align*}
\]

From (6) and (7), we obtain

\[
\begin{align*}
\frac{\partial \theta_1}{\partial \theta_4} & = \frac{i_3 \sin (\theta_1 + \theta_2 + \theta_4)}{l_1 \sin \theta_2} \\
\frac{\partial \theta_2}{\partial \theta_4} & = \frac{i_3 \sin (\theta_1 + \theta_4)}{l_2 \sin \theta_2} \\
\frac{\partial \theta_3}{\partial \theta_4} & = \frac{l_3 \sin (\theta_1 + \theta_2 + \theta_4)}{l_1 \sin \theta_2}.
\end{align*}
\]

Because \( \theta_1 \) is the measured output, for convenience, the dynamics is expressed in terms of \( \theta_1 \) as

\[
\mathcal{M}(\theta) \ddot{\theta}_1 + \mathcal{C}(\theta, \dot{\theta}_1) \dot{\theta}_1 + \mathcal{K}(\theta, \dot{\theta}_1) = \tau
\]

with

\[
\begin{align*}
\mathcal{M}(\theta) & := D(\theta) \frac{\partial \theta_4}{\partial \theta_1} \\
\mathcal{C}(\theta, \dot{\theta}_1) & := \frac{1}{2} \hat{D}(\theta) \frac{\partial \theta_4}{\partial \theta_1} + D(\theta) \frac{d}{dt} \frac{\partial \theta_4}{\partial \theta_1} \\
\mathcal{K}(\theta) & := \hat{K}(\dot{\theta}) \frac{\partial \theta_4}{\partial \theta_1}.
\end{align*}
\]

Note that \( \theta_2 \) and \( \theta_4 \) are functions of \( \theta_1 \), thus in the following, the above defined functions can alteratively be denoted by \( \mathcal{M}(\theta_1), \mathcal{C}(\theta_1, \dot{\theta}_1), \) and \( \mathcal{K}(\theta_1) \).
III. TRACK-FOLLOWING CONTROL VIA LOW-FREQUENCY NONLINEARITY PRECOMPENSATION

A. Approximate Linear Systems

For the flexible model (1), define $\omega = \dot{\vartheta}$, we can get the state space representation

$$\begin{cases}
\dot{\vartheta} = \omega \\
\dot{\omega} = M^{-1}(\vartheta) \left[-C(\theta, \omega)\omega - \frac{\partial V(\theta)}{\partial \theta} - K\omega + b\tau\right]
\end{cases}$$

(9)

where $\eta$ is the output. Note that at the equilibrium point $(\bar{\vartheta}, 0)$, there holds that $C(\bar{\vartheta}, 0) = 0$, $\partial V(\theta) / \partial \theta |_{\bar{\vartheta}} = 0$, and

$$\frac{\partial}{\partial \theta} |_{(\bar{\vartheta}, 0)} \left[C(\theta, \omega)\omega\right] = 0, \quad \frac{\partial}{\partial \omega} |_{(\bar{\vartheta}, 0)} \left[C(\theta, \omega)\omega\right] = 0.$$

We get the linearization of (9) at the equilibrium point $(\bar{\vartheta}, 0)$, as

$$\begin{cases}
\dot{\omega} = \omega \\
\dot{\eta} = M^{-1}(\bar{\vartheta}) \left[-\frac{\partial V(\theta)}{\partial \theta} |_{\bar{\vartheta}} \right] \omega - K\omega + b\tau
\end{cases}$$

(10)

where $e = \theta - \bar{\theta}$ and $e_1 = \theta_1 - \bar{\theta}_1$. The dynamics of (10) can be equivalently represented by

$$M(\bar{\theta}) \ddot{e} + K e + \left[\frac{\partial^2 V(\theta)}{\partial \theta^2} \right]_{\bar{\theta}} e = b\tau.$$  

(11)

Denote $e = [1, 0, 0]^T$, then the transfer function from $\tau$ to $e_1$ is

$$G_{\bar{\theta}}(s) = c^T \left[s^2 M(\bar{\theta}) + sK + \frac{\partial^2 V(\theta)}{\partial \theta^2} \right]_{\bar{\theta}}^{-1} b.$$  

(12)

$G_{\bar{\theta}}(s)$ is a transfer function with relative degree 2 satisfying

$$z_0(\bar{\theta}) := \lim_{s \to \infty} s^2 G_{\bar{\theta}}(s) = c^T M^{-1}(\bar{\theta}) b.$$  

Note that any point $\bar{\theta}$ satisfying $\lambda_x(\bar{\theta}) = 0$ and $\lambda_y(\bar{\theta}) = 0$ is a minimum point of the potential energy function $V(\theta)$, i.e.,

$$0 = V(\bar{\theta}) \leq V(\theta), \quad \forall \theta.$$

But on the other hand, any such $\bar{\theta}$ is not an isolated minimum point of $V(\theta)$ which means that $V(\theta)$ has no strict minimum point, i.e., there exists no $\bar{\theta}$ such that

$$V(\bar{\theta}) < V(\theta), \quad \forall \theta \neq \bar{\theta}.$$  

This implies that the Hessian matrix of $V(\theta)$ is positive semidefinite rather than positive definite. Thus we have

$$\det \left[\left[\frac{\partial^2}{\partial \theta^2} V(\theta)\right]_{\bar{\theta}}\right] = 0$$  

(13)

which implies that $G_{\bar{\theta}}(s)$ has a pole located at the origin no matter what $\bar{\theta}$ is. Since $G_{\bar{\theta}}(s)$ is of even order, there must be another real pole, denoted by $-p_1(\bar{\theta})$, which depends on the track position $\bar{\theta}$. The two springs at the driving joint $A_1$ used to model the joint flexibility implies that two resonance modes should appear in $G_{\bar{\theta}}(s)$, i.e., the other two pairs of poles are complex. Based on this observation, $G_{\bar{\theta}}(s)$ can be decomposed as

$$G_{\bar{\theta}}(s) = P_1(\bar{\theta}) \Sigma_{\bar{\theta}}(s)$$  

(14)

where

$$P_1(\bar{\theta}) = \frac{z_0(\bar{\theta})}{s^2 + p_1(\bar{\theta}) s},$$

$$\Sigma_{\bar{\theta}}(s) = \prod_{i=1}^{2} \frac{s^2 + 2\xi_1(\bar{\theta}) \omega_1(\bar{\theta}) s + \omega_1(\bar{\theta})^2}{s^2 + 2\xi_2(\bar{\theta}) \omega_1(\bar{\theta}) s + \omega_1(\bar{\theta})^2}.$$  

$P_1(\bar{\theta})$ and $\Sigma_{\bar{\theta}}(s)$ are called the low frequency dynamics and the high frequency dynamics at $\bar{\theta}$, respectively. Note that at any equilibrium point $(\bar{\theta}, 0)$, there holds $(x, y) = (0, 0)$ and $\bar{\theta}$ is subjected to constraints (6), i.e., $\lambda_x(\bar{\theta}) = 0$ and $\lambda_y(\bar{\theta}) = 0$ with $\lambda_x$ and $\lambda_y$ given in (7). Thus the coefficients $z_0(\bar{\theta}_1), p_0(\bar{\theta}_1), \xi_1(\bar{\theta}_1), \xi_2(\bar{\theta}_1), z_1(\bar{\theta}_1),$ and $p_1(\bar{\theta}_1)$ are actually functions of $\bar{\theta}_1$, denoted by $z_0(\bar{\theta}_1), p_0(\bar{\theta}_1), \xi_1(\bar{\theta}_1), z_1(\bar{\theta}_1),$ and $p_1(\bar{\theta}_1)$, respectively.

It is worth pointing out that the steady-state gain of the high frequency dynamics $\Sigma_{\bar{\theta}}(s)$ is not necessarily equal to 1, i.e.,

$$\Sigma_{\bar{\theta}}(0) = \prod_{i=1}^{2} \frac{z_2(\bar{\theta}_1)}{p_2(\bar{\theta}_1)} \neq 1.$$  

Thus the low-frequency dynamics $P_1(\bar{\theta})$ does not completely coincide with the approximate linear systems of the rigid model (8). The approximate linear system of the rigid model (8) at $\bar{\theta}_1$ can be easily calculated as

$$M(\bar{\theta}_1) \ddot{\dot{e}} + K(\bar{\theta}_1) \dot{\dot{e}} = \tau$$  

(15)

whose transfer function is

$$G_{\bar{\theta}_1}(s) = \frac{M^{-1}(\bar{\theta}_1)}{s + M^{-1}(\bar{\theta}_1) K(\bar{\theta}_1)}.$$  

(16)

The transfer functions $G_{\bar{\theta}}(s)$ and $G_{\bar{\theta}_1}(s)$ are related by

$$G_{\bar{\theta}}(s) = P_1(s) \Sigma_{\bar{\theta}}(0).$$  

(17)

Thus we have

$$\begin{cases}
M^{-1}(\bar{\theta}) = z_0(\bar{\theta}) \Sigma_{\bar{\theta}}(0) \\
M^{-1}(\bar{\theta}) K(\bar{\theta}) = p_0(\bar{\theta})
\end{cases}.$$  

(18)
B. Prefilter Design

The prefilter is designed based on the approximate linear system (16) of the rigid model. Choose any \( \vec{\theta}_1 \) within the whole range of \( \theta_1 \) as the nominal value. Then for any target position \( \vec{\theta}_1 \), a simple first order prefilter can be constructed as

\[
C_{\vec{\theta}_1}(s) = \frac{M(\vec{\theta}_1)s + K(\vec{\theta}_1)}{M(\vec{\theta}_1)s + K(\vec{\theta}_1)}.
\]  

(19)

This prefilter is connected with the plant in series as depicted in Fig. 4. It is straightforward to check that the approximate linear system of the series connection of the prefilter \( C_{\vec{\theta}_1}(s) \) and the plant at any track position \( \vec{\theta}_1 \), denoted by \( \tilde{G}_{\vec{\theta}_1}(s) \), is identical to the series connection of the prefilter and the approximate linear system of the plant. Thus, by (14), (16), and (17), we have

\[
\tilde{G}_{\vec{\theta}_1}(s) = G_{\vec{\theta}_1}(s)C_{\vec{\theta}_1}(s)
\]

\[
= P_{\vec{\theta}_1}(s)\Sigma_{\vec{\theta}_1}(s)C_{\vec{\theta}_1}(s)
\]

\[
- P_{\vec{\theta}_1}(s)\Sigma_{\vec{\theta}_1}(s)C_{\vec{\theta}_1}(s) \cdot \Sigma_{\theta}^{-1}(0)\Sigma_{\theta}(s)
\]

\[
= \frac{M^{-1}(\vec{\theta}_1)}{s + M^{-1}(\vec{\theta}_1)K(\vec{\theta}_1)} \cdot \left[ \Sigma_{\theta}^{-1}(0)\Sigma_{\theta}(s) \right]
\]

\[
= \frac{M^{-1}(\vec{\theta}_1)}{s + M^{-1}(\vec{\theta}_1)K(\vec{\theta}_1)} \cdot \left[ \Sigma_{\theta}^{-1}(0)\Sigma_{\theta}(s) \right]
\]

where \( \Gamma_{\theta}(s) := \Sigma_{\theta}^{-1}(0)\Sigma_{\theta}(s) \). The first part \( G_{\vec{\theta}_1}(s) \) is the approximate linear system of the rigid model at the nominal track position \( \vec{\theta}_1 \) and does not depend on the set-point \( \vec{\theta}_1 \). The second part \( \Gamma_{\theta}(s) \) depends on the track position and satisfies \( \Gamma_{\theta}(0) = 1 \).

The only information needed in the prefilter (19) is the target track position \( \vec{\theta}_1 \) and the parameters \( M(\vec{\theta}_1) \) and \( K(\vec{\theta}_1) \) can be calculated online without effort, thus it is easy for implementation. With this prefilter, the approximate linear systems of the compensated system at different track positions have the same low-frequency dynamics. Many existing track-following design methods can then be applied to the nominal model

\[
\tilde{G}_{\vec{\theta}_1}(s) = G_{\vec{\theta}_1}(s)\Gamma_{\theta}(s)
\]

of the compensated plant by viewing the deviations in high-frequency resonances as uncertainties.

C. \( H_\infty \) Loop Shaping

In this paper, the \( H_\infty \) loop shaping method is used for the nominal compensated plant \( \tilde{G}_{\vec{\theta}_1}(s) \), as shown in Fig. 5. The weighing function \( W_{\theta}(z) \) is chosen to shape the sensitivity function. Then a controller \( C(z) \) is to be designed such that the closed-loop system is stable and the \( H_\infty \)-norm from \( w \) to \( z \) is less than a given \( \gamma \). For the detailed \( H_\infty \) loop shaping design, we refer to [11]–[13].

Suppose that the state-space realizations of \( \tilde{G}_{\vec{\theta}_1}(z) \) (i.e., the discrete-time version of \( \tilde{G}_{\vec{\theta}_1}(s) \)) and \( W_{\theta}(z) \) are, respectively, \( (A_p, B_p, C_p, D_p) \) and \( (A_s, B_s, C_s, D_s) \). Then the state-space representation of the augmented system in Fig. 5 is

\[
\begin{align*}
x(k+1) &= Ax(k) + Buw(k) + Bu(u(k) + y(k) = Cy(k) + Dyw(k) + Dyu(k) \tag{20}
\end{align*}
\]

with

\[
A = \begin{bmatrix} A_p & 0 \\ B_s C_p & A_s \end{bmatrix}, \quad B_w = \begin{bmatrix} 0 \\ B_s \end{bmatrix}, \quad B_u = \begin{bmatrix} H_p \\ B_s D_p \end{bmatrix} \quad C_y = \begin{bmatrix} C_p \\ 0 \end{bmatrix}, \quad D_{yw} = D_s, \quad D_{yu} = D_s D_p
\]

Proposition 1: [12] Suppose that the plant \( \tilde{G}_{\vec{\theta}_1}(z) \) is strictly proper, i.e., \( D_p = 0 \). Then for a given \( \gamma > 0 \), there exists a dynamic output feedback controller \( C(z) \) that stabilizes (20) and the \( H_\infty \)-norm from \( w \) to \( z \) is less than \( \gamma \) if there exist matrices \( X, I, Y, F, Q, R, S, J \) and symmetric matrices \( P \) and \( H \) such that the linear matrix inequality (LMI) (21) shown at the bottom of the next page. A feasible controller can be constructed as

\[
\begin{align*}
x_{\theta}(k+1) &= Ax_{\theta}(k) + Bu_{\theta}(k) + H_y y(k) \\
u(k) &= C_x x_{\theta} + D_y y(k) \tag{22}
\end{align*}
\]

with

\[
D_c = R \quad C_c = (I - RC_y X) U^{-1}, \tag{23}
\]

\[
\begin{bmatrix}
P & J & AX + BuL & A + BuRC_y & Bu + BuRDyw & 0 \\
* & H & Q & YA + FC_y & YB_u + FDyw & 0 \\
* & * & X + X^T & I + ST - J & 0 & X^T C_{2n} + I^T D_{2n}^T \\
* & * & * & Y + Y^T & 0 & C_{2n}^2 + C_{2n}^2 R^T D_{2n}^T \\
* & * & * & * & 0 & \gamma^T I \\
* & * & * & * & * & 0
\end{bmatrix} > 0. \tag{21}
\]
Fig. 8. Bode plots of the approximate linear systems of the compensated plant at different positions (discretized version).

\begin{align}
B_c &= V^{-1}(F - YB_aR) \\
A_c &= V^{-1}[Q - Y(A + B_aD_cCy)X - VBB_cX]U^{-1} \\
&\quad - V^{-1}YB_aC_c
\end{align}

(25)

(26)

Remark 1: The requirement \(D_{yu} = 0\) can be removed if the LMI (21) admits a set of solution such that \(\det(I + D_cD_{yu}) \neq 0\). Refer to [14]. In this case, a feasible controller can be constructed as

\begin{equation}
\begin{cases}
x_c(k + 1) = A_c x_c(k) + B_c y(k) \\
u(k) = C_c x_c + D_c y(k)
\end{cases}
\end{equation}

(27)

with

\begin{align}
A_c &= A_c - B_c D_{yu}^{-1}(I + D_cD_{yu})^{-1}C_c \\
B_c &= B_c - B_c D_{yu}^{-1}(I + D_cD_{yu})^{-1}D_c \\
C_c &= (I + D_cD_{yu})^{-1}C_c \\
D_c &= (I + D_cD_{yu})^{-1}D_c
\end{align}

(23)–(26)

where \(A_c, B_c, C_c, \) and \(D_c\) are given in (23)–(26).

IV. SIMULATION RESULTS

The parameters for simulation are given in Table I. With these parameters, the range of \(\theta_1\) is approximately from \(-0.3353\) to \(0.5754\) radian. Fig. 6 gives the Bode plots of \(G_h^s(s)\) at different track positions. We can see that the approximate linear systems at different positions differ significantly. The prefilter is designed according to (19) and discretized with sampling rate \(2.5 \times 10^{-5}\) s. Fig. 7 gives the Bode’s plots of the prefilter at the different track positions. The Bode’s plots of the approximate linear systems of the compensated plant are given in Fig. 8. From this figure, one sees that the prefilter makes the low-frequency dynamics of the approximate linear systems at different track positions almost the same.

By choosing \(\theta_1^0 = 0.4193\) radian as the nominal value, the nominal compensated plant used for \(H_\infty\) design (discrete-time version) is given by

\begin{equation}
G_{\hat{p}}(z) = \frac{1.127 \times 10^{-5}(z + 1)}{(z - 1)(z - 0.9818)} \times \frac{z^2 + 0.7829z + 0.9978}{z^2 + 0.7055z + 0.9976} \times \frac{z^2 + 0.2321z + 0.9905}{z^2 + 0.216z + 0.9896}
\end{equation}

The weighing function \(W_s\) is chosen as

\begin{equation}
W_s(z) = \frac{0.537z^2 - 0.6143z + 0.1757}{z^2 - 1.994z + 0.9937}
\end{equation}

\(1/W_s(z)\) roughly gives the required shape of the sensitivity whose magnitude-frequency characteristic is given in Fig. 11. With this \(W_s(z)\), the \(H_\infty\) controller \(C\) is given by (28) shown at the bottom of the page. Fig. 9 gives the open-loop Bode plot at the nominal track position \(\theta_1^0\) under the proposed controller and prefilter. It shows that the proposed control provides a gain margin of 8 dB at 9.8 kHz and a phase margin of 45.6 degrees at 4.45 kHz. The proposed controller and prefilter are applied to the original nonlinear plant at different track positions and the corresponding open loop Bode plots, sensitivity and output responses are given in Fig. 10, Fig. 11, and Fig. 12, respectively. These simulation results show that the prefilter effectively compensated for the main nonlinearities and the proposed controller leads to similar performance under different track-following situations within the range \(\theta_1 \in [0.0370, 0.5229]\).

V. CONCLUDING REMARKS

In this paper, the track-following control design of a four-bar structured hard disk drive system was investigated. A simple prefilter was proposed to compensate for the low-frequency

\begin{equation}
C(z) = \frac{-520.7z^8 + 961.6z^7 - 1104z^6 + 1787z^5 - 1652z^4 + 1065z^3 - 933.2z^2 + 392.7z + 4.081}{z^8 - 0.1627z^7 + 0.0476z^6 - 1.136z^5 - 0.7114z^4 + 0.1351z^3 + 0.2373z^2 + 0.5867z + 0.08025}.
\end{equation}

(28)
nonlinearities so that the compensated plant have almost the same linear approximation subjected to different high-frequency resonances. With this prefilter, $H_{\infty}$ loop shaping was applied to the nominal plant to obtain a track-following controller which applies to almost all the track-following situations. Simulation results showed the validity of the proposed method.

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