Spectral Gaps of Schrödinger Operators and Diffusion Operators on Abstract Wiener Spaces

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Abstract

In this paper we extend the spectral gap comparison theorem of Andrews and Clutterbuck [J. Amer. Math. Soc. 24 (2011), no. 3, 899–916] to the infinite dimensional setting. More precisely, we prove that the spectral gap of the Schrödinger operator $-\mathcal{L} + V$ (\(\mathcal{L}\) is the Ornstein–Uhlenbeck operator) on the abstract Wiener space is greater than that of the one dimensional operator $-\frac{d^2}{ds^2} + \tilde{V}(s)$, provided that $\tilde{V}$ is a modulus of convexity for $V$. Similar result is established for the diffusion operator $-\mathcal{L} + \nabla F \cdot \nabla$.

Keywords: Spectral gap, Schrödinger operator, abstract Wiener space, min-max principle, Malliavin calculus.

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1 Introduction

In this paper, we will compare the spectral gap of a self-adjoint Schrödinger operator on abstract Wiener space with that on one dimensional Gaussian space. This work is a natural extension of the famous \textit{fundamental gap conjecture}, solved by Andrews and Clutterbuck [1] most recently, which gave an optimal lower bound of $\lambda_1 - \lambda_0$, the distance between the first two Dirichlet eigenvalues of a Schrödinger operator $-\Delta + V$ on a bounded uniformly convex domain $\Omega$ with a weakly convex potential $V$. For the original conjecture with its literature, we refer to [1] and references therein.

We briefly recall Andrews and Clutterbuck’s arguments. They introduced the notion of \textit{modulus of convexity} for $V$ which plays an important role in their proof. Namely, an even function $\tilde{V} \in C^1((-\frac{D}{2}, \frac{D}{2}) \ (D = \text{diam}(\Omega))$ is called a modulus of convexity for $V$ if

$$
(\nabla V(x) - \nabla V(y)) \cdot \frac{x-y}{|x-y|} \geq 2\tilde{V}'\left(\frac{|x-y|}{2}\right), \ \forall x \neq y \in \Omega.
$$

Under this assumption, Andrews and Clutterbuck proved an estimate of the \textit{modulus of log-concavity} for the ground state $\phi_0$ (i.e. the eigenfunction associated to the first eigenvalue $\lambda_0$):

$$
(\nabla \log \phi_0(x) - \nabla \log \phi_0(y)) \cdot \frac{x-y}{|x-y|} \leq 2(\log \tilde{\phi}_0)'\left(\frac{|x-y|}{2}\right),
$$

where $\tilde{\phi}_0$ is the ground state of $-\frac{d^2}{ds^2} + \tilde{V}(s)$ with Dirichlet boundary condition. Moreover, let $u_i = e^{-\lambda_i t}\phi_i$, $i = 0, 1$, be two solutions to the Dirichlet heat equation $\frac{\partial u}{\partial t} = \Delta u - Vu$ on
Consider \( \Omega \times [0, \infty) \), then one can get the modulus of continuity for the ground transformation \( v = \frac{u_1 - u_0}{u_0} \), i.e.

\[
v(x, t) - v(y, t) \leq e^{-(\lambda_1 - \lambda_0)t} \text{osc}_X \phi_1 - \phi_0 \leq 2C e^{-(\lambda_1 - \lambda_0)t},
\]

where \( \lambda_0 \) and \( \lambda_1 \) are the first two Dirichlet eigenvalues of \( \frac{d^2}{dx^2} + \tilde{V}(s) \). Consequently, letting \( t \to \infty \) gives us the comparison

\[
\lambda_1 - \lambda_0 \geq \lambda_1 - \lambda_0.
\]

This is the beautiful strategy in [1] to solve the conjecture, which works for smooth potentials and compact domains.

Our purpose is to generalize these results to the infinite dimensional setting. Let \((\mathcal{W}, \mathcal{H}, \mu)\) be an abstract Wiener space and \( \mathcal{L}^* \) the Ornstein–Uhlenbeck operator on \( \mathcal{W} \) associated to the symmetric Dirichlet form \( \mathcal{E}_s(f, f) = (f, -\mathcal{L}_s f) \) with domain \( \mathcal{D}[\mathcal{E}_s] = \mathcal{D}^2_1(\mathcal{W}, \mu) \) (i.e. \( f \in L^2(\mathcal{W}, \mu) \)) with its Malliavin derivative \( \nabla f \in L^2(\mathcal{W}, \mathcal{H}) \). Let \( V \in \mathcal{D}_p^2(\mathcal{W}, \mu) \) for some \( p > 1 \) be a potential satisfying the KLMN condition (see Theorem X.17 in Reed and Simon [6]), i.e. there exist some \( 0 < a < 1 \) and \( b > 0 \) such that

\[
\int V - f^2 \, d\mu \leq a \left( \mathcal{E}_s(f, f) + \int V^+ f^2 \, d\mu \right) + b \int f^2 \, d\mu,
\]

where \( V^+ = \max\{V, 0\} \) and \( V^- = -\min\{V, 0\} \). Thus one can define \( -\mathcal{L}^* = -\mathcal{L} + V \) to be a self-adjoint Schrödinger operator bounded from below, which is associated to the symmetric Dirichlet form

\[
\mathcal{E}_s(f, f) = \int (|\nabla f|^2 + V f^2) \, d\mu
\]

with domain

\[
\mathcal{D}[\mathcal{E}] = \{ f \in \mathcal{D}^2_1(\mathcal{W}, \mu) : V f^2 \text{ is } L^1\}-\text{integrable} \}.
\]

It is well known that there are two kinds of equivalent min-max principles for general self-adjoint operator \( H \) bounded from below, that is \( \mu_i = \lambda_i \) for all \( i \geq 0 \) which are defined as follows (by convention, \( \varphi_0, \varphi_1, \ldots, \varphi_i \) are linearly independent and \( [\varphi_0, \varphi_1, \ldots, \varphi_i]^\perp \) denotes the orthogonal completion of \( \text{span}\{\varphi_0, \varphi_1, \ldots, \varphi_i\} \)):

1. \( \mu_i = \sup_{\varphi_0, \varphi_1, \ldots, \varphi_i \in D[H]} \inf_{\varphi \in [\varphi_0, \varphi_1, \ldots, \varphi_i]^\perp} \langle \varphi, H\varphi \rangle \)
2. \( \lambda_i = \inf_{\varphi_0, \varphi_1, \ldots, \varphi_i \in D[H]} \sup_{\varphi \in [\varphi_0, \varphi_1, \ldots, \varphi_i]^\perp} \langle \varphi, H\varphi \rangle \)

Here we take \( H = -\mathcal{L}^* \), and actually we can replace \( \langle \varphi, H\varphi \rangle \) by \( \mathcal{E}(\varphi, \varphi) \) and \( \mathcal{D}[H] \) by \( \mathcal{D}[\mathcal{E}] \) or \( \mathcal{D}[\mathcal{E}] \cap L^\infty(\mathcal{W}, \mu) \) equivalently. When \( \lambda_1 - \lambda_0 > 0 \), we say the spectral gap exists. At this time, \( \lambda_0 \) has to be an isolated eigenvalue due to [6, Theorem XIII.1], and the associated eigenfunctions are called ground states. When \( \lambda_0 \) is of multiplicity one, we denote by \( \phi_0 \) for the \( L^2 \)-normalized ground state.

Correspondingly, let \( \mathcal{L}_s = \frac{d^2}{dx^2} - \frac{a}{d} \) be the one-dimensional Ornstein–Uhlenbeck operator on \( \mathbb{R}^1 \) with respect to the Gaussian measure \( d\gamma_1 = (4\pi)^{-\frac{1}{2}} \exp(-\frac{x^2}{4}) \, ds \). Let \( \tilde{V} \in C^1(\mathbb{R}^1) \cap L^1(\mathbb{R}^1, \gamma_1) \) be a symmetric potential satisfying the KLMN condition too. Then \( -\mathcal{L} = -\mathcal{L}_s + \tilde{V} \) is bounded from below, and the associated Dirichlet form is

\[
\tilde{\mathcal{E}}(f, f) = \int (|f'|^2 + \tilde{V} f^2) \, d\gamma_1
\]

\[2\]
with domain
\[ \mathcal{D}[\hat{\mathcal{E}}] = \{ f \in \mathbb{D}_2^1(\mathbb{R}^1, \gamma) : \hat{V} f^2 \text{ is } L^1 \text{-integrable} \} . \]

For convenience, a tilde will be added to all notations relative to \( \hat{\mathcal{L}}_* \) and \( \hat{V} \).

Let us state the main results in this paper. In the following, \( \langle \cdot, \cdot \rangle_{\mathbb{H}} \) denotes the inner product in the Cameron–Martin space \( \mathbb{H} \), and \( \| \cdot \|_{\mathbb{H}} \) the \( L^2 \)-norm.

**Theorem 1.1.** Suppose for almost all \( w \in \mathbb{W} \) and every \( h \in \mathbb{H} \) with \( h \neq 0 \)
\[ \left\langle \nabla V(w + h) - \nabla V(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \geq 2\hat{V}' \left( \frac{\|h\|_{\mathbb{H}}}{2} \right) . \] (1.2)

Then there exists a comparison
\[ \lambda_1 - \lambda_0 \geq \hat{\lambda}_1 - \hat{\lambda}_0 . \]

Hence, the existence of the spectral gap of \( -\mathcal{L} \) on Wiener space can sometimes be reduced to one dimensional case. According to [1], \( \hat{V} \) is a modulus of convexity for \( V \). However, \( V \) doesn’t need to be convex at all.

The next result gives the modulus of log-concavity for \( \phi_0 \).

**Theorem 1.2.** Assume the same condition as in Theorem 1.1 and the gap \( \hat{\lambda}_1 - \hat{\lambda}_0 > 0 \). Then \( -\mathcal{L} \) and \( -\hat{\mathcal{L}} \) have a unique ground state respectively. Moreover, for almost all \( w \in \mathbb{W} \) and every \( h \in \mathbb{H} \) with \( h \neq 0 \),
\[ \left\langle \nabla \log \phi_0(w + h) - \nabla \log \phi_0(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \leq 2(\log \phi_0)' \left( \frac{\|h\|_{\mathbb{H}}}{2} \right) . \]

Section 2 and 3 will be devoted to the above theorems. In Section 4, we consider the diffusion operator \( -\mathcal{L} = -\mathcal{L}_* + \nabla F \cdot \nabla \) on the Wiener space and we want to compare its spectral gap with the one dimensional operator \( -\hat{\mathcal{L}} = -\frac{d^2}{dx^2} + (s + \omega'(s))\frac{d}{dx} \). Although this kind of diffusion operator can be transformed to the Schrödinger type operator and their spectrum coincide with each other (see Remark 4.1 for the details), the expression for the potential function \( V \) is a little complicated, hence it seems inappropriate to derive the gap comparison of diffusion operators from that of the transformed Schrödinger type operators. We shall directly establish the comparison theorem for spectral gaps of diffusion operators in Section 4. The main result (see Theorem 4.13) we obtained there is analogous to Theorem 1.1.

## 2 Comparison over \( \mathbb{R}^n \)

In this section, we will discuss the spectral gaps comparison and ground states approximation for a Schrödinger operator on \( \mathbb{R}^n \) with a weakly derivable potential, as the preliminary versions of Theorem 1.1 and 1.2 on finite dimensional spaces.

We denote by \( \hat{\mathcal{L}}_* = \Delta - x \cdot \nabla \) the Ornstein–Uhlenbeck operator on \( \mathbb{R}^n \) with respect to the Gaussian measure \( d\gamma_n = (4\pi)^{-\frac{n}{2}} \exp(-|x|^2) dx \). Let \( \hat{V} \in \mathbb{D}_1((\mathbb{R}^n, \gamma_n)) \) satisfy the KLMN condition. Then one can define \( -\hat{\mathcal{L}} = -\hat{\mathcal{L}}_* + \hat{V} \) to be a self-adjoint Schrödinger operator bounded from below, which is associated to
\[ \hat{\mathcal{E}}(f, f) = \int \left( |\nabla f|^2 + \hat{V} f^2 \right) d\gamma_n \]
with domain
\[ \mathcal{D}[\hat{\mathcal{E}}] = \{ f \in \mathbb{D}_2^1(\mathbb{R}^n, \gamma) : \hat{V} f^2 \text{ is } L^1 \text{-integrable} \} . \]

For convenience, a bar will be added to all the relative notation to \( \hat{\mathcal{L}}_* \) and \( \hat{V} \). With a slight abuse of notation, we still denote by \( \langle \cdot, \cdot \rangle \) the inner product and \( \| \cdot \| \) the norm of \( L^2(\mathbb{R}^n, \gamma_n) \).
2.1 Smooth potential

In this subsection, let’s assume $\bar{V}$ is smooth.

**Proposition 2.1.** Suppose $\bar{V} \in C^\infty(\mathbb{R}^n)$ such that for any $x, y \in \mathbb{R}^n$ with $x \neq y$,

$$\nabla \bar{V}(x) - \nabla \bar{V}(y)) \cdot \frac{x - y}{|x - y|} \geq 2\bar{V}' \left( \frac{|x - y|}{2} \right).$$

Then the spectral gap of $-\bar{L}$ satisfies

$$\tilde{\lambda}_1 - \tilde{\lambda}_0 \geq \tilde{\lambda}_1 - \tilde{\lambda}_0.$$

The strategy of proof is to give the comparison of spectral gaps for Schrödinger operators restricted on arbitrary ball, and then prove the approximation of min-max expressions when the radius tends to infinity.

Let $\Omega$ be any open ball in $\mathbb{R}^n$ of diameter $D$, and correspondingly $I = [-\frac{D}{2}, \frac{D}{2}]$. According to [1, Proposition 3.1], we consider the restricted Schrödinger operator $-\bar{L}_\Omega = -\bar{L}_* + \bar{V}_\Omega$ on $\Omega$, and the parabolic equation with Dirichlet boundary condition

$$\begin{cases}
\frac{\partial u}{\partial t} = \Delta u - x \cdot \nabla u - \bar{V}u, & \text{on } \Omega \times [0, \infty); \\
u = 0, & \text{on } \partial \Omega \times [0, \infty).
\end{cases}$$

Suppose that $u_0$ and $u_1$ are the solutions corresponding to the first two Dirichlet eigenvalues $\bar{\lambda}_0(\Omega)$ and $\bar{\lambda}_1(\Omega)$ of $-\bar{L}_\Omega$. Put $v = \frac{u_1}{u_0}$, called the *ground state transformation*, which solves the Neumann heat equation:

$$\begin{cases}
\frac{\partial v}{\partial t} - \Delta v - (2\nabla \log u_0 - x) \cdot \nabla v = 0, & \text{on } \Omega \times [0, \infty); \\
D_v v = 0, & \text{on } \partial \Omega \times [0, \infty).
\end{cases}$$

Define a potential function $\bar{W}(x) = \bar{V}(x) + \frac{1}{4}|x|^2 - \frac{D^2}{2}$, which satisfies

$$(-\Delta + \bar{W})(u_0 e^{-\frac{1}{4}|x|^2}) = \bar{\lambda}_0(\Omega)(u_0 e^{-\frac{1}{4}|x|^2}),$$

in other words, $u_0 e^{-\frac{1}{4}|x|^2}$ is a ground state of the Schrödinger operator $-\Delta + \bar{W}$. The same theory holds for $\bar{\lambda}_0(I)$, $\bar{\lambda}_1(I)$, $\bar{v} = \frac{u_1}{u_0}$ and $\bar{W}(s) = \bar{V}(s) + \frac{1}{4}s^2 - \frac{1}{2}$ on $I$.

By (2.1), $\bar{W}$ has a modulus of convexity $\bar{W}$ such that for any $x \neq y$,

$$\nabla \bar{W}(x) - \nabla \bar{W}(y)) \cdot \frac{x - y}{|x - y|} \geq 2\bar{W}' \left( \frac{|x - y|}{2} \right).$$

Applying [1, Theorem 4.1] to $\bar{W}$ with $\bar{W}$ yields that the vector field

$$X = 2\nabla \log u_0 - x = 2\nabla \log(u_0 e^{-\frac{1}{4}|x|^2})$$

has a modulus of contraction $\omega(s) = 2\log(\bar{u}_0 e^{-\frac{1}{4}s^2})$ on $[0, \frac{D}{2}]$ satisfying for any $x \neq y$,

$$(X(x) - X(y)) \cdot \frac{x - y}{|x - y|} \leq 2\omega \left( \frac{|x - y|}{2} \right).$$

Hence, applying Theorem 2.1 and Proposition 3.2 in [1] to the above $X$, we obtain

$$\tilde{\lambda}_1(\Omega) - \tilde{\lambda}_0(\Omega) \geq \tilde{\lambda}_1(I) - \tilde{\lambda}_0(I).$$

(2.2)

Now, we need to prove the approximation in (2.2) when the diameter $D$ goes to infinity.
Lemma 2.2. Let $\Omega_k$ be a sequence of open balls centered at 0 of radius $k$. Define $\bar{L}_{\Omega_k}$ as follows: for any $f \in C_0(\Omega_k) \cap C^2(\Omega_k)$,
\[ \bar{L}_{\Omega_k}f(x) = \bar{L}f(x), \quad x \in \Omega_k; \]
and for $i \geq 0$, define
\[ \bar{\lambda}_i(\Omega_k) = \inf_{\varphi_0, \varphi_1, \ldots, \varphi_i \in \mathcal{D}[\bar{L}_{\Omega_k}]} \sup_{||\varphi|| = 1, \varphi \in \text{span}\{\varphi_0, \varphi_1, \ldots, \varphi_i\}} (\varphi, -\bar{L}_{\Omega_k}\varphi). \]
Then $\bar{\lambda}_i(\Omega_k) \to \bar{\lambda}_i$ ($k \to \infty$).

Proof. Given $\varepsilon > 0$. Since $\mathcal{D}[\bar{L}_{\Omega_k}]$ is the completion of $C_c^\infty(\Omega_k)$, there exists an $i+1$ dimensional subspace $L \subset C_c^\infty(\Omega_k)$ such that
\[ \bar{\lambda}_i(\Omega_k) + \varepsilon > \sup_{\varphi \in L, ||\varphi|| = 1} (\varphi, -\bar{L}_{\Omega_k}\varphi). \]
It follows from the min-max principle that
\[ \bar{\lambda}_i(\Omega_k) + \varepsilon > \bar{\lambda}_i. \]
Hence, we obtain $\lim_{k \to \infty} \bar{\lambda}_i(\Omega_k) \geq \bar{\lambda}_i$.

On the other hand, for any $\varepsilon > 0$, there exists $\{\varphi_0, \varphi_1, \ldots, \varphi_i\} \subset \mathcal{D}[\bar{L}] \cap L^\infty$ such that
\[ \bar{\lambda}_i + \varepsilon \geq \sup_{||\varphi|| = 1, \varphi \in \text{span}\{\varphi_0, \varphi_1, \ldots, \varphi_i\}} (\varphi, -\bar{L}_{\Omega_k}\varphi). \]
Thus we can find some $k_0$ such that for all $k \geq k_0$ and all $\varphi \in \text{span}\{\varphi_0, \varphi_1, \ldots, \varphi_i\}$ with $||\varphi|| = 1$, it holds
\[ \int_{\Omega_k} \varphi \cdot (-\bar{L}\varphi) \, d\gamma_n \leq \varepsilon, \quad ||\varphi||_\infty \int_{\Omega_k} |\nabla|^2 \, d\gamma_n \leq \varepsilon \quad \text{and} \quad \int_{\Omega_k} \varphi^2 \, d\gamma_n \leq \varepsilon. \]
Choose some $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $0 \leq \rho \leq 1$, $\rho|_{\Omega_0} = 1$ and $|\nabla\rho| \leq 1$, thus
\[ \left| \int_{\mathbb{R}^n} \varphi \cdot (-\bar{L}\varphi) \, d\gamma_n - \int_{\mathbb{R}^n} \rho \varphi \cdot (-\bar{L}(\rho \varphi)) \, d\gamma_n \right| \leq 10 \varepsilon. \]
Note that $1 - \varepsilon \leq ||\rho \varphi|| \leq 1$, it follows
\[ \bar{\lambda}_i + \varepsilon \geq (1 - \varepsilon)^2 \sup_{||\varphi|| = 1, \varphi \in \mathcal{D}[\bar{L}], \varphi \in \text{span}\{\varphi_0, \rho \varphi_1, \ldots, \rho \varphi_i\}} (\varphi, -\bar{L}_{\Omega_k}\varphi) \geq (1 - \varepsilon)^2 \bar{\lambda}_i(\Omega_k). \]
Hence, we obtain $\lambda_i \geq \lim_{k \to \infty} \lambda_i(\Omega_k)$. The proof is completed.

Now we go back to Proposition 2.1.

Proof. Combine (2.2) and Lemma 2.2.
2.2 Weakly derivable potential

In this subsection, let’s assume \( \tilde{V} \) is weakly derivable.

**Proposition 2.3.** Suppose that \( \tilde{V} \in D^1_t(\mathbb{R}^n, \gamma_n) \) satisfies for almost every \( x, y \in \mathbb{R}^n \) with \( x \neq y \),

\[
(\nabla \tilde{V}(x) - \nabla \tilde{V}(y)) \cdot \frac{x - y}{|x - y|} \geq 2 \tilde{V}' \left( \frac{|x - y|}{2} \right).
\]

Then the spectral gap of \( -\tilde{L} \) satisfies

\[
\lambda_1 - \lambda_0 \geq \hat{\lambda}_1 - \hat{\lambda}_0.
\]

Since \( \nabla \tilde{V} \) exists in the sense of distribution, we have to use certain mollifier. Here we choose the Ornstein–Uhlenbeck semigroup \( (P_t)_{t \geq 0} \). Put \( c_t = e^{-2t} \) (which is useful to (2.6) below), \( \tilde{V}_t(x) = c_t P_t \tilde{V}(x) \) and \( \tilde{V}_t(s) = c_t \tilde{V}(e^{-t}s) \), thus by straightforward calculation,

\[
(\nabla \tilde{V}_t(x) - \nabla \tilde{V}_t(y)) \cdot \frac{x - y}{|x - y|} = c_t e^{-t} (\nabla \tilde{V}(x) - P_t(\nabla \tilde{V})(y)) \cdot \frac{x - y}{|x - y|}
\]

\[
= c_t e^{-t} \int \left( \nabla \tilde{V}(e^{-t}x + \sqrt{1 - e^{-2t}} \xi) - \nabla \tilde{V}(e^{-t}y + \sqrt{1 - e^{-2t}} \xi) \right) \, d\gamma_n(\xi) \cdot \frac{x - y}{|x - y|}
\]

\[
\geq 2c_t e^{-t} \tilde{V}' \left( e^{-t} \frac{|x - y|}{2} \right) = 2 \tilde{V}' \left( \frac{|x - y|}{2} \right),
\]

which means \( \tilde{V}_t \) is a modulus of convexity for \( \tilde{V}_t \).

Before applying Proposition 2.1 to \( \tilde{V}_t \) and \( \tilde{V} \) directly, the KLMN condition has to be verified for two families of quadratic forms

\[
\tilde{E}_t(\varphi, \varphi) = (\varphi, (-\tilde{L}_* + \tilde{V}_t)\varphi), \quad \tilde{E}_t(\varphi, \varphi) = (\varphi, (-\tilde{L}_* + \tilde{V}_t)\varphi).
\]

In other words, the KLMN condition should be stable under certain perturbations. We point out that the following lemma doesn’t use the modulus of convexity (2.3).

**Lemma 2.4.** Assume that \( -\tilde{L}_* + \tilde{V} \) and \( -\tilde{L}_* + \tilde{V} \) satisfy the KLMN condition. For small \( t > 0 \), both \( \tilde{a}(-\tilde{L}_* + \tilde{V}_t^+) \) and \( \tilde{a}(-\tilde{L}_* + \tilde{V}_t^-) \) are uniformly bounded from below.

**Proof.** If the first statement is not true, then there exists a sequence of \( f_t \in C^1_b(\mathbb{R}^n) \) such that \( \|f_t\| = 1 \) and

\[
\tilde{a} \int (|\nabla f_t|^2 + \tilde{V}_t^+ f_t^2) \, d\gamma_n - \int \tilde{V}_t^- f_t^2 \, d\gamma_n = -\tilde{C}_t \to -\infty, \quad \text{when } t \to 0,
\]

where \( \tilde{V}_t^+ = c_t P_t \tilde{V}_t^+ \) and \( \tilde{V}_t^- = c_t P_t \tilde{V}_t^- \).

By straightforward calculation,

\[
\int \tilde{V}_t^\pm f_t^2 \, d\gamma_n = \int \tilde{V}^\pm \cdot [c_t P_t(f_t^2)] \, d\gamma_n.
\]

Put \( g_t = \sqrt{c_t P_t(f_t^2)} + \varepsilon \) for some \( \varepsilon > 0 \), which satisfies \( \|g_t\| = c_t + \varepsilon \) and

\[
\int |\nabla g_t|^2 \, d\gamma_n = c_t e^{-2t} \int \frac{|P_t(f_t \cdot \nabla f_t)|^2}{P_t(f_t^2)} + \varepsilon \, d\gamma_n \leq \int |\nabla f_t|^2 \, d\gamma_n.
\]
It follows that
\[ \tilde{a} \int ((|\nabla g|)^2 + \tilde{V}^+ g_t^2) \, d\gamma_n - \int \tilde{V}^- g_t^2 \, d\gamma_n \leq -\tilde{C}_t + \varepsilon \int |\tilde{V}| \, d\gamma_n \rightarrow -\infty, \]
which contradicts to the KLMN condition for \(-\tilde{\mathcal{L}}_* + \tilde{V}\).

Similarly, if the second statement is not true, there exists another sequence of \(u_t \in C^1_b(\mathbb{R})\) such that \(||u_t|| = 1\) and
\[ \tilde{a} \int (|u_t'|^2 + \tilde{V}_t^+ u_t^2) \, d\gamma_n - \int \tilde{V}_t^- u_t^2 \, d\gamma_n = -\tilde{C}_t \rightarrow -\infty, \quad \text{when } t \to 0, \]
where \(\tilde{V}_t^+(s) = c_t \tilde{V}^+(e^{-t}s)\) and \(\tilde{V}_t^-(s) = c_t \tilde{V}^-(e^{-t}s)\).

By straightforward calculation,
\[ \int \tilde{V}_t^+ u_t^2 \, d\gamma_n = \int \tilde{V}^+(s) \cdot \left( c_t u_t^2(e^t s) \frac{d\gamma_n(e^t s)}{d\gamma_n(s)} \right) \, d\gamma_n(s). \]

Put \(v_t(s) = u_t(e^t s) \sqrt{c_t \frac{d\gamma_n(e^t s)}{d\gamma_n(s)}}\), which satisfies \(||v_t||^2 = c_t\) and
\[ \int |v_t'|^2 \, d\gamma_n = c_t \left\{ e^{2t} \int |u_t'|^2 \, d\gamma_n + \frac{e^{2t} - 1}{2} ||u_t||^2 - \frac{(e^t - e^{-t})^2}{4} ||s u_t||^2 - \frac{1 - e^{-2t}}{2} ||s u_t||^2 \right\} \]
\[ \leq \int |u_t'|^2 \, d\gamma_n + \frac{1 - e^{-2t}}{2}. \tag{2.6} \]

It follows that
\[ \tilde{a} \int (|v_t'|^2 + \tilde{V}_t^+ v_t^2) \, d\gamma_n - \int \tilde{V}_t^- v_t^2 \, d\gamma_n \leq -\tilde{C}_t + \frac{1 - e^{-2t}}{2} \rightarrow -\infty, \]
which contradicts to the KLMN condition for \(-\tilde{\mathcal{L}}_* + \tilde{V}\).

By Lemma 2.4, the Dirichlet forms \(\tilde{\mathcal{E}}_t(\cdot, \cdot)\) and \(\tilde{\mathcal{E}}_t(\cdot, \cdot)\) (see (2.4)) are well defined and bounded from below. Recall (2.3), let’s substitute \(\tilde{V}\) and \(\tilde{V}\) in Proposition 2.1 to \(\tilde{V}_t\) and \(\tilde{V}_t\) to get
\[ \tilde{\lambda}_{1, t} - \tilde{\lambda}_{0, t} \geq \tilde{\lambda}_{1, t} - \tilde{\lambda}_{0, t}, \tag{2.7} \]
where we still use the min-max principle to define \(\tilde{\lambda}_{i, t}\) corresponding to \(-\tilde{\mathcal{L}}_* + \tilde{V}_t\) and \(\tilde{\lambda}_{i, t}\) to \(-\tilde{\mathcal{L}}_* + \tilde{V}_t\) for every \(i \geq 0\).

Now, we stand at the position to prove the approximation of \(\tilde{\lambda}_{i, t}\) and \(\tilde{\lambda}_{i, t}\) as \(t \to 0\). The next two lemmas don’t require (2.3) too.

**Lemma 2.5.** For every \(i \geq 0\), \(\tilde{\lambda}_i \geq \lim_{t \to 0} \tilde{\lambda}_{i, t}\). Especially, \(\tilde{\lambda}_0 = \lim_{t \to 0} \tilde{\lambda}_{0, t}\).

**Proof.** For any \(\varepsilon > 0\), there exists an \(i + 1\) dimensional subspace \(L\) spanned by an orthogonal family \(\{\varphi_0, \varphi_1, \ldots, \varphi_i\} \subset C_0^\infty(\mathbb{R}^n)\) such that
\[ \tilde{\lambda}_i + \varepsilon \geq \sup_{\varphi \in L, ||\varphi|| = 1} \tilde{\mathcal{E}}(\varphi, \varphi). \]

Since \(L\) is of finite dimension, the dominated convergence theorem yields a \(t_0 > 0\) such that for all \(t \in (0, t_0)\) and \(\varphi \in L\) with \(||\varphi|| = 1\),
\[ |\tilde{\mathcal{E}}(\varphi, \varphi) - \tilde{\mathcal{E}}_t(\varphi, \varphi)| \leq \int \left| \tilde{V} (\varphi^2 - c_t P_t(\varphi^2)) \right| \, d\gamma_n \leq \varepsilon, \]
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which implies due to the min-max principle

$$\bar{\lambda}_i + 2\varepsilon \geq \bar{\lambda}_{i,t}.$$  

Hence, we obtain $$\bar{\lambda}_i \geq \lim_{t \to 0} \bar{\lambda}_{i,t}.$$  

Furthermore, for any $$\varphi \in D[\hat{\mathcal{E}}] \cap L^\infty$$ with $$\|\varphi\| = 1$$, we have

$$\tilde{\mathcal{E}}_t(\varphi, \varphi) = \int |\nabla \varphi|^2 d\gamma_n + \int \bar{V} \cdot c_t P_t(\varphi^2) d\gamma_n.$$  

Analogous to (2.5), put $$\varphi_t = \sqrt{c_t P_t(\varphi^2) + \varepsilon}$$ such that $$\|\varphi_t\|^2 = c_t + \varepsilon$$ and

$$\int |\nabla \varphi_t|^2 d\gamma_n \leq \int |\nabla \varphi|^2 d\gamma_n, \quad \int \bar{V} \varphi_t^2 d\gamma_n = \int \bar{V} \varphi^2 d\gamma_n + \varepsilon \int \bar{V} d\gamma_n,$$

which implies that

$$\tilde{\mathcal{E}}(\varphi_t, \varphi_t) \leq \tilde{\mathcal{E}}_t(\varphi, \varphi) + \varepsilon \int \bar{V} d\gamma_n.$$  

Then the min-max principle for $$i = 0$$ gives

$$(c_t + \varepsilon)^2 \bar{\lambda}_0 \leq \bar{\lambda}_{0,t} + \varepsilon \int \bar{V} d\gamma_n.$$  

Hence, we obtain $$\bar{\lambda}_0 \leq \lim_{t \to 0} \bar{\lambda}_{0,t}$$. The proof is completed. \hfill \Box

**Lemma 2.6.** For every $$i \geq 0$$, $$\bar{\lambda}_{i,t}$$ converges to $$\bar{\lambda}_i$$ as $$t \to 0$$.

**Proof.** For any $$\varepsilon > 0$$, take some $$i+1$$ dimensional subspace $$L$$ generated by an orthogonal family $$\{\varphi_0, \varphi_1, \ldots, \varphi_i\} \subset C_c^\infty(\mathbb{R})$$ such that

$$\bar{\lambda}_i + \varepsilon \geq \sup_{\varphi \in L, \|\varphi\| = 1} \tilde{\mathcal{E}}(\varphi, \varphi).$$

For any $$\varphi \in L$$ with $$\|\varphi\| = 1$$, we have by straightforward calculation that

$$\tilde{\mathcal{E}}(\varphi, \varphi) = \int |\varphi'|^2 d\gamma_1 + \int \tilde{V} \varphi^2 d\gamma_1$$

$$= \int |\varphi'|^2 d\gamma_1 + \int \tilde{V}_i(s) \cdot c_i^{-1} \varphi(e^{-t} s) \frac{d\gamma_1(e^{-t} s)}{d\gamma_1(s)} d\gamma_1(s).$$

Analogous to (2.6) (but a bit different here), put $$\varphi_t(s) = \varphi(e^{-t} s) \sqrt{c_t^{-1} \cdot \frac{d\gamma_1(e^{-t} s)}{d\gamma_1(s)}}$$, which satisfies $$\|\varphi_t\|^2 = c_t^{-1}$$ and

$$\int |\varphi_t'|^2 d\gamma_1 = c_t^{-1} \left\{ e^{-2t} \int |\varphi'|^2 d\gamma_1 + \frac{e^{-2t} - 1}{2} \|\varphi\|^2 - \frac{(e^t - e^{-t})^2}{4} \|\varphi\|^2 - \frac{1 - e^{-2t}}{2} \|\varphi\|^2 \right\}.$$  

Since $$L$$ is of finite dimension, there exists some $$t_0 > 0$$ such that for all $$t \in (0, t_0)$$

$$\int |\varphi_t'|^2 d\gamma_1 \leq \int |\varphi'|^2 d\gamma_1 + \varepsilon,$$

which implies that

$$\tilde{\mathcal{E}}_t(\varphi_t, \varphi_t) \leq \tilde{\mathcal{E}}(\varphi, \varphi) + \varepsilon.$$
It follows from the min-max principle
\[ c_t^{-1} \lambda_{i,t} \leq \lambda_i + 2\varepsilon. \]

Hence, we obtain \( \lim_{t \to 0} \tilde{\lambda}_{i,t} \leq \lambda_i \).

On the other hand, let \( L_t \) be some \( i + 1 \) dimensional subspace generated by an orthogonal family \( \{ \psi_0, \psi_1, \ldots, \psi_i \} \subset C_c^\infty(\mathbb{R}) \) such that
\[ \lambda_{i,t} + \varepsilon \geq \sup_{\psi \in L_t, \|\psi\|=1} \tilde{\mathcal{E}}_t(\psi, \psi). \]

For any \( \psi \in L_t \) with \( \|\psi\| = 1 \)
\[ \tilde{\mathcal{E}}_t(\psi, \psi) = \int |\psi'|^2 d\gamma_1 + \int \tilde{V}_t \psi^2 d\gamma_1 \]
\[ = \int |\psi'|^2 d\gamma_1 + \int \tilde{V}(s) \cdot c_t \psi(e^t s)^2 \frac{d\gamma_1(e^t s)}{d\gamma_1(s)} d\gamma_1(s). \]

As in (2.6), put \( \psi_t(s) = \psi(e^t s) \sqrt{\frac{d\gamma_1(e^t s)}{d\gamma_1(s)}} \), which satisfies \( \|\psi_t\|^2 = c_t \) and
\[ \int |\psi_t'|^2 d\gamma_1 \leq \int |\psi'|^2 d\gamma_1 + 1 - e^{-2t}. \]

It follows that
\[ \tilde{\mathcal{E}}(\psi, \psi_t) \leq \tilde{\mathcal{E}}_t(\psi, \psi) + \frac{1 - e^{-2t}}{2}, \]
which implies
\[ c_t \lambda_i \leq \tilde{\lambda}_{i,t} + \frac{1 - e^{-2t}}{2} + \varepsilon. \]

Hence, we obtain \( \tilde{\lambda}_i \leq \lim_{t \to 0} \tilde{\lambda}_{i,t} \). The proof is completed. \( \Box \)

Now we go back to Proposition 2.3.

Proof. Combine (2.7) and Lemmas 2.5-2.6. \( \Box \)

2.3 Modulus of ground state

In this subsection, we will give the modulus of log-concavity for the ground state of \(-\tilde{L}\).

Proposition 2.7. Suppose \( \lambda_1 - \lambda_0 > 0 \). Then \(-\tilde{L}\) (resp. \(-\tilde{L}\)) has a unique ground state \( \tilde{\phi}_0 \) (resp. \( \tilde{\phi}_0 \)). Moreover, for almost every \( x \neq y \),
\[ (\nabla \log \tilde{\phi}_0(x) - \nabla \log \tilde{\phi}_0(y)) \cdot \frac{x - y}{|x - y|} \leq 2(\log \tilde{\phi}_0)'\left(\frac{|x - y|}{2}\right). \]

For ease of notation, let \( H = -A + W \) be a nonnegative self-adjoint Schrödinger operator and \( \phi \) a ground state with \( H \phi = \phi \phi \). Let \( H_k = -A_k + W_k \) be a sequence of Schrödinger operator bounded from below and \( \phi_k \) a ground state with \( H_k \phi_k = \phi_k \phi_k \). Denote by \( \mathcal{E}_H, \mathcal{E}_A \) and \( \mathcal{E}_{H_k} \) the Dirichlet forms associated to \( H \), \(-A\) and \( H_k \) respectively. Recall previous subsections, we will actually deal with three cases:

1. \( H = -\tilde{L}_* + \tilde{V}, \ A = \tilde{L}_* \), and \( H_k = (-\tilde{L}_* + \tilde{V})|_{\Omega_k} \), here \( C_c^\infty(\mathbb{R}^n) \) is dense in \( \mathcal{D}[H] \) and \( C_c^\infty(\Omega_k) \) dense in \( \mathcal{D}[H_k] \);
2. \( H = -\mathcal{L}_s + \tilde{V}, \ A = \tilde{\mathcal{L}}_s, \) and \( H_k = -\mathcal{L}_s + c_k P_k \tilde{V}, \) here \( C^\infty_c(\mathbb{R}^n) \) is dense in both \( D[H] \) and \( D[H_k]; \)

3. \( H = -\mathcal{L}_s + \tilde{V}, \ A = \tilde{\mathcal{L}}_s, \) and \( H_k = -\mathcal{L}_s + c_k \tilde{V}(e^{-t_k}s), \) here \( C^\infty_c(\mathbb{R}^1) \) is dense in both \( D[H] \) and \( D[H_k]. \)

Note that we have shown \( \lim_{k \to \infty} \varrho_k = \varrho. \)

The following lemma gives the ground states approximation in a unified way for the above three cases, which doesn’t use the modulus of convexity (2.3) too.

**Lemma 2.8.** Suppose that

1. The spectral gaps of \( H \) and \( H_k \) are all greater than \( \delta_{\text{gap}} > 0; \)

2. As ground states, \( \phi \) and \( \phi_k \) are all of multiplicity one;

3. there exists \( \kappa > 0 \) such that \(-A + (1 + \kappa)W \) and \(-A_k + (1 + \kappa)W_k \) are uniformly bounded from below.

Then \( \|\phi_k - \phi\| \to 0 \) and \( E_A[\phi_k - \phi] \to 0 \) when \( k \to \infty. \)

**Proof.** The proof is divided into three steps. By convention, \( \phi \) and \( \phi_k \) are normalized in \( L^2. \)

**Step 1.** Given \( \varepsilon > 0. \) Take \( \varphi \in C^\infty_c(\mathbb{R}^n) \) such that \( \varphi \geq 0, \|\varphi\| = 1 \) and

\[
(\phi - \varphi, H(\phi - \varphi)) + \|\phi - \varphi\| \leq \varepsilon. \tag{2.8}
\]

Let \( \{\phi_k\}^\perp \) be the orthogonal complement of \( \{\phi_k\}. \) Consider the orthogonal decomposition \( \varphi - \phi_k = w_k + \alpha_k \phi_k, \) where \( w_k \in \{\phi_k\}^\perp. \) It follows

\[
1 = \|\varphi\|^2 = \|w_k\|^2 + (1 + \alpha_k)^2\|\phi_k\|^2 \geq (1 + \alpha_k)^2. \tag{2.9}
\]

Put \( \eta_k = H_k \varphi - \varrho \varphi \) (for big \( k \) such that \( \varphi \in D[H_k] \)), we have by straightforward calculation

\[
H_k w_k = (\varrho - \varrho_k) \varphi + \varrho_k w_k + \eta_k,
\]

which implies

\[
(\varphi, H_k w_k) = (\varrho - \varrho_k)(\varphi, \varphi) + \varrho_k(\varphi, w_k) + (\varphi, \eta_k). \tag{2.10}
\]

For this \( \varepsilon \) fixed above, there exists \( k_0 \) such that \( |\varrho - \varrho_k| \leq \varepsilon \) for all \( k \geq k_0. \) Here we assume

\[
|(\varphi, \eta_k)| \leq \varepsilon, \tag{2.11}
\]

which will be verified in the third step. We can rewrite (2.10) as

\[
\varrho - \varrho_k + (\varphi, \eta_k) = (\varphi, H_k w_k - \varrho_k w_k)
\]

\[
= (\varphi - \varrho_k - \alpha_k \varphi_k, H_k w_k - \varrho_k w_k)
\]

\[
= (w_k, H_k w_k) - \varrho_k \|w_k\|^2
\]

\[
\geq (\delta_{\text{gap}} - \varepsilon) \|w_k\|^2. \tag{2.12}
\]

The last inequality comes from \( w_k \in \{\phi_k\}^\perp \) and \( \phi_k \) is the unique ground state of \( H_k. \) Hence \( \|w_k\| = O(\varepsilon) \) by \( \varrho_k \to \varrho \) and Assumption (2.11).
Recall (2.9), it follows \((1 + \alpha_k)^2 = 1 + O(\varepsilon)\), that is, \(\alpha_k = O(\varepsilon)\) or \(\alpha_k = -2 + O(\varepsilon)\). Assume there is a subsequence \(\alpha_{k_i} = -2 + O(\varepsilon)\), then for big \(i\), the nonnegativity of ground states gives
\[
w_{k_i} = \varphi - (1 + \alpha_{k_i})\phi_{k_i} \geq \frac{1}{2}\phi_{k_i},
\]
which is in contradiction with \(\|w_k\| = O(\varepsilon)\). So we have \(\alpha_k = O(\varepsilon)\) and
\[
\|\phi_k - \varphi\|^2 = \|\phi_k - \varphi\|^2 + O(\varepsilon) = \|w_k\|^2 + \alpha_k^2\|\phi_k\|^2 + O(\varepsilon) = O(\varepsilon).
\]
Hence, we obtain \(\|\phi_k - \varphi\|^2 \to 0\).

**Step 2.** If there exists \(\kappa > 0\) such that \(-A + (1 + \kappa)W\) is bounded from below, we can take some \(K \geq 0\) such that \(-A + (1 + \kappa)W + K\) is a nonnegative self-adjoint operator. Let \(C = \kappa^{-1}\), formally we have
\[
-A \leq (C + 1)(-A + W) + CK,
\]
which means for any \(u\),
\[
\mathcal{E}_A[u] \leq (C + 1)\mathcal{E}_W[u] + CK\|u\|^2.
\]
Similarly we have \(\mathcal{E}_A[u] \leq (C + 1)\mathcal{E}_W[u] + CK\|u\|^2\) too.

According to Step 1, since \((w_k, H_kw_k) = O(\varepsilon), \|\phi_k - \varphi\| = O(\varepsilon)\) and \(\alpha_k \to 0\), we have \(\mathcal{E}_W[\phi_k - \varphi] = O(\varepsilon)\), which implies \(\mathcal{E}_A[\phi_k - \varphi] = O(\varepsilon)\). Hence, we obtain \(\mathcal{E}_A[\phi_k - \varphi] \to 0\).

**Step 3.** Now we stand at the position to verify Assumption (2.11) for all the three explicit expressions of \(H_k\).

**Case (i).** Recall subsection 2.1 for the smooth potential \(\bar{V}\), let
\[
H_k = (-\bar{L}_s + \bar{V})_{|\Omega_k},
\]
which means we only consider the operator restricted on the open ball \(\Omega_k\).

By (2.8), for \(k\) so big that \(\text{supp}(\varphi) \subset \Omega_k\), one has
\[
|\langle \varphi, \eta_k \rangle| = |\varphi(\varphi) - \varphi, H_k\varphi| = |\varphi(\varphi) - \varphi, H\varphi| \\
\leq |\varphi(\varphi) - \mathcal{E}_H[\varphi]| + |\mathcal{E}_H[\varphi] - \mathcal{E}_H[\varphi]| \leq 2\varepsilon.
\]
So Assumption (2.11) is fulfilled.

**Case (ii).** Recall subsection 2.2 for \(\bar{V}\), let
\[
H_k = -\bar{L}_s + c_{t_k}P_{t_k}\bar{V},
\]
such that \(c_{t_k}P_{t_k}\bar{V}\) becomes a smooth potential.

We estimate by (2.8) and dominated convergence theorem for big \(k\)
\[
|\langle \varphi, \eta_k \rangle| = |\varphi(\varphi) - \int (|\nabla \varphi|^2 + c_{t_k}P_{t_k}\bar{V} \cdot \varphi^2) \, d\gamma_n| \\
\leq |\varphi(\varphi) - \mathcal{E}_H[\varphi]| + \int (\bar{V}(\varphi^2 - c_{t_k}P_{t_k}(\varphi^2)) \, d\gamma_n| \leq 2\varepsilon.
\]

**Case (iii).** Recall subsection 2.2 for \(\bar{V}_t(s) = c_t\bar{V}(e^{-t}s), let\)
\[
H_k = -\bar{L}_s + \bar{V}_t.
\]

Put $\varphi_{i_k}(s) = \varphi(e^{ks})\sqrt{e^{ks} \frac{d^2 \varphi(e^{ks})}{d s^2}}$, we estimate by using (2.8) and the dominated convergence theorem for big $k$

$$|(\varphi, \eta_k)| = |\varphi(\varphi, \varphi) - \int (|\varphi'|^2 + \tilde{V}_{i_k} \cdot \varphi^2) \, d \gamma_1|$$

$$\leq |\varphi(\varphi, \varphi) - \mathcal{E}_H[\varphi]| + \left| \int \tilde{V}(\varphi^2 - \varphi_{i_k}^2) \, d \gamma_1 \right| \leq 2\varepsilon.$$ 

The proof is completed.

Now, we go back to Proposition 2.7.

Proof. Due to [1, Theorem 1.5] for the modulus of log-concavity for ground state on bounded domain associated to smooth potential, we can apply Lemma 2.8 to prove Proposition 2.7 by ground states approximation. So it is enough to verify all the conditions in Lemma 2.8. In fact, Condition 1 is obviously fulfilled by setting $\delta_{\text{gap}} = \lambda_1 - \lambda_0$.


Condition 3 comes also from the KLMN condition by taking $0 < \kappa < \bar{a}^{-1} - 1$ such that

$$\bar{a} \left( \mathcal{E}_s(f, f) + (1 + \kappa) \int \tilde{V} f^2 \, d \gamma_n \right) + \bar{b} \int f^2 \, d \gamma_n \geq |1 - \bar{a}(1 + \kappa)| \int \tilde{V}^{-1} f^2 \, d \gamma_n \geq 0,$$

and Lemma 2.4 has shown the stability of KLMN condition under the perturbations $\tilde{V}_{i_k}$. The consideration for $\tilde{V}$ is similar.

3 Comparison over $\mathbb{W}$

Now, we will compare the spectral gaps between $-\mathcal{L}$ on $(\mathbb{W}, H, \mu)$ and $-\mathcal{L}$ on $(\mathbb{R}^1, \gamma_1)$ via the finite dimensional approximantation.

First of all, we recall some basic notations and facts on abstract Wiener spaces. The readers are referred to [3, 5] for details. $H$ is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$, which is called the Cameron–Martin space, and we denote by

$$\mathcal{F}_H = \{ X \subset H : X \text{ is a finite dimensional linear subspace} \}.$$ 

$\mathbb{W}$ is a completion of $H$ under a radonifying norm $\| \cdot \|_\mathbb{W}$ satisfying that $\| \cdot \|_H \leq C \| \cdot \|_\mathbb{W}$ and for any $\varepsilon > 0$, there exists $X \in \mathcal{F}_H$ such that for any $Y \in \mathcal{F}_H$ orthogonal to $X$,

$$\gamma_Y(y \in Y : \| y \|_\mathbb{W} \geq \varepsilon) \leq \varepsilon,$$

where $\gamma_Y$ is the standard Gaussian measure on $Y$. Then there is an inclusion relation $\mathbb{W}^* \subset H^* = H \subset \mathbb{W}$, and we can take an orthogonal basis of $H$ as $\{ e_i \in \mathbb{W}^* \}_{i \geq 1}$.

For $n \geq 1$, let $X_n = \text{span}\{ e_1, e_2, \ldots, e_n \}$ be a $n$-dimensional linear subspace such that $H = \bigcup_{n \geq 1} X_n$. For every $X_n$, there exist the direct sum $\mathbb{W} = X_n \oplus Y_n$ and measure decomposition $\mu = \gamma_n \otimes \mu_n$. Let $P_{X_n}$ be the orthogonal projection from $H$ onto $X_n$, and $\pi_n : \mathbb{W} \rightarrow X_n$ its extension to $\mathbb{W}$, that is, $\pi_n(w) = \sum_{i=1}^n e_i(w)e_i$. Then

$$\pi_n(w + h) - \pi_n(w) = P_{X_n}h. \quad (3.1)$$

For any $F \in L^1(\mathbb{W}, \mu)$, define $\mathbb{E}^{X_n}(F)$ to be the $L^1$ conditional expectation of $F$ on the sub-Borel algebra generated by $\pi_n$, and there exists $f : X_n \rightarrow \mathbb{R}$ such that $f \circ \pi_n = \mathbb{E}^{X_n}(F)$. Furthermore, $\mathbb{E}^{X_n}(F)$ converges to $F$ in the $L^1$-norm.
For $F \in L^p(W, \mu)$ with $p > 1$, it is called Malliavin derivable, denoted by $F \in D^p(W, \mu)$, if there exists $\nabla F \in L^p(W, \mathbb{H})$ such that for any $h \in \mathbb{H}$,

$$(\nabla F(w), h)_{\mathbb{H}} = D_h F(w) := \lim_{\varepsilon \to 0} \frac{F(w + \varepsilon h) - F(w)}{\varepsilon}.$$ 

Here the reason for $p > 1$ is that it is convenient to define the shift operator $\tau_h$ by the Cameron–Martin theorem (i.e. the integral transformation on $W$)

$$\tau_h F(w) := F(w + h) \in L^p(W, \mu),$$

where $L^p(W, \mu) = \mathbb{H} \cap p' < p L^p(W, \mu)$. Moreover, we have the formula (cf. [3, Proposition 3.3.8])

$$\nabla f(x) = P_{X_n} \left( \int_{Y_n} \nabla F(x,y) \, d\mu_n(y) \right), \quad x = \pi_n(w).$$

(3.2)

Let $S(\mathbb{R}^n)$ be the Schwartz test functions on $\mathbb{R}^n$. We denote by

$$\text{Cylin}(W) = \{ F : W \to \mathbb{R} | \text{there exist } n \geq 1 \text{ and } f \in S(\mathbb{R}^n) \text{ such that } F = f \circ \pi_n \}$$

the cylindrical Wiener functionals, which is dense in $L^p(W, \mu)$ and also the Sobolev spaces $D^q_1(W, \mu)$ for $q > 1$, see for instance [3, Theorem 3.3.9].

Recall our setting in the beginning, we suppose the potential $V \in D^p(W, \mu)$ ($p > 1$), and the Schrödinger operator $H = -\Delta + V$ is bounded from below, together with a sequence of $\lambda_i$ defined by the min-max principle. For simplicity, we identify $X_n$ with $\mathbb{R}^n$ and denote $V_n \circ \pi_n = E_{X_n}(V)$.

### 3.1 Approximation of projections

This subsection is devoted to the proof of Theorem 1.1.

**Lemma 3.1.** Suppose for almost all $w \in W$ and every $h \in \mathbb{H}$ with $h \neq 0$, it holds

$$\left( \nabla V(w + h) - \nabla V(w), \frac{h}{\|h\|} \right)_{\mathbb{H}} \geq 2 \tilde{V}' \left( \frac{\|h\|}{2} \right).$$

Then $\tilde{V}$ is also a modulus of convexity of $V_n$, that is for a.e. $x, y \in \mathbb{R}^n$ with $x \neq y$,

$$(\nabla V_n(x) - \nabla V_n(y)) \cdot \frac{x - y}{|x - y|} \geq 2 \tilde{V}' \left( \frac{|x - y|}{2} \right).$$

**Proof.** Due to (3.1) and (3.2). \hfill \square

Corresponding to $\lambda_i$, define by the min-max principle for $H_n = -\Delta + x \cdot \nabla + V_n$ on $L^2(\mathbb{R}^n, \gamma_n)$ that

$$\lambda_{i,n} = \inf_{\varphi_0, \varphi_1, \ldots, \varphi_i \in D[H_n]} \sup_{\|\varphi\| = 1, \varphi \in \text{span}\{\varphi_0, \varphi_1, \ldots, \varphi_i\}} (\varphi, H_n \varphi).$$

Since $V_n$ is weakly derivable and has a modulus of convexity as $\tilde{V}$, we can use Proposition 2.3 to get

$$\lambda_{1,n} - \lambda_{0,n} \geq \tilde{\lambda}_1 - \tilde{\lambda}_0.$$ 

In the following we prove the approximation $\lambda_{i,n} \to \lambda_i$. 

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Lemma 3.2. Cylindrical functions are dense in domain $D[\mathcal{E}]$.

Proof. It is sufficient to show $\text{Cylin}(\mathcal{W})$ is dense in $D[\mathcal{E}] \cap L^\infty(\mathcal{W}, \mu)$. This is true because $\text{Cylin}(\mathcal{W})$ is dense in $D^1(\mathcal{W}, \mu)$ and $L^\infty(\mathcal{W}, \mu)$, and $V$ is $L^1$-integrable. 

\[ \text{Lemma 3.3. For every } i \geq 0, \lambda_{i,n} \text{ converges to } \lambda_i \text{ as } n \to \infty. \]

Proof. The inclusion $X_n \subset X_{n+1} \cdots \subset \mathbb{H}$ implies $D[H_n] \subset D[H_{n+1}] \cdots \subset D[H]$, and thus $\lambda_{i,n}$ is decreasing with $\lim_{n \to \infty} \lambda_{i,n} \geq \lambda_i$.

On the other hand, for any $\varepsilon > 0$, Lemma 3.2 yields an $(i + 1)$-dimensional linear subspace $L$ generated by normalized cylindrical functions $\{\varphi_0, \varphi_1, \ldots, \varphi_i\}$ such that, for any $\varphi \in L$ with $\|\varphi\| = 1$,

$$\lambda_i + \varepsilon \geq \int |\nabla \varphi|^2 \, d\mu + \int V\varphi^2 \, d\mu.$$ 

So for big $n$, we have $\varphi = E^{X_n}(\varphi)$ and $\varphi^2 = E^{X_n}(\varphi^2)$. Denote by $\psi \circ \pi_n = \varphi$, then

$$\int |\nabla \varphi|^2 \, d\mu + \int V\varphi^2 \, d\mu = \int |\nabla \psi|^2 \, d\gamma_n + \int V_n\psi^2 \, d\gamma_n,$$

which implies by the min-max principle that

$$\lambda_i + \varepsilon \geq \lambda_{i,n}.$$ 

Hence $\lim_{n \to \infty} \lambda_{i,n} = \lambda_i$. 

Combining the above lemmas, we get the proof of Theorem 1.1.

3.2 Approximation of ground states

In this subsection, we will prove Theorem 1.2. Let $\phi_0$ be a ground state of $-\mathcal{L}$ with $-\mathcal{L}\phi_0 = \lambda_0\phi_0$. Correspondingly, let $\phi_{0,n}$ be a ground state of $H_n = -\Delta + x \cdot \nabla + V_n$ on $L^2(\mathbb{R}^n, \gamma_n)$ such that $H_n\phi_{0,n} = \lambda_{0,n}\phi_{0,n}$. The following lemma gives the approximation of $\phi_{0,n}$, which has the same spirit as Lemma 2.8.

Lemma 3.4. Suppose that

1. $\lambda_1 - \lambda_0 > 0$;
2. $\phi_0$ and $\phi_{0,n}$ are all of multiplicity one;
3. there exists $\kappa > 0$ such that $-\mathcal{L} + (1 + \kappa)V$ is bounded from below.

Then $\phi_{0,n} \circ \pi_n$ converges to $\phi_0$ in the norm $\| \cdot \|_W$, and $\nabla(\phi_{0,n} \circ \pi_n)$ to $\nabla \phi$ as well.

Proof. By convention, all the ground states are normalized in $L^2$.

Given $\varepsilon > 0$. Since $\phi_0$ is nonnegative, choose $\varphi_0 \in D[\mathcal{L}] \cap L^\infty(\mathcal{W}, \mu)$ such that $\varphi_0 \geq 0$, $\|\varphi_0\|_W = 1$ and

$$\|\phi_0 - \varphi_0, -\mathcal{L}(\phi_0 - \varphi_0)\|_W + \|\phi_0 - \varphi_0\|^2_W \leq \varepsilon.$$ 

Denote $\varphi_{0,n} \circ \pi_n = E^{X_n}(\varphi_0)$, we have

$$\|\varphi_{0,n}\| \to 1, \quad (\varphi_{0,n}, H_n \varphi_{0,n}) \to (\varphi_0, -\mathcal{L}\phi_0) = \lambda_0 + O(\varepsilon), \quad (3.3)$$

where we use the dominated convergence theorem to get $\int V_n\varphi_{0,n}^2 \, d\gamma_n \to \int V\phi_0^2 \, d\mu$. 

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Consider the orthogonal decomposition
\[ \varphi_{0,n} - \phi_{0,n} = w_n + \alpha_n \phi_{0,n}, \]
where \( w_n \) belongs to the orthogonal complement \( \{ \phi_{0,n} \}^\perp \) in \( L^2(\mathbb{R}^n, \gamma_n) \). It follows
\[ \| \varphi_{0,n} \|^2 = \| w_n \|^2 + (1 + \alpha_n)^2 \| \phi_{0,n} \|^2 \geq (1 + \alpha_n)^2. \] (3.4)

Denote \( \eta_n = H_n \varphi_{0,n} - \lambda_0 \varphi_{0,n} \), we have by straightforward computation
\[
H_n(w_n + \alpha_n \phi_{0,n}) = H_n \varphi_{0,n} - \lambda_0 \phi_{0,n} \\
= (\lambda_0 - \lambda_0, n) \varphi_{0,n} + \lambda_0 (\varphi_{0,n} - \phi_{0,n}) + \eta_n \\
= (\lambda_0 - \lambda_0, n) \varphi_{0,n} + \lambda_0 (w_n + \alpha_n \phi_{0,n}) + \eta_n,
\]
which implies
\[ H_n w_n = (\lambda_0 - \lambda_0, n) \varphi_{0,n} + \lambda_0, n w_n + \eta_n. \]

It follows that
\[ (\varphi_{0,n}, H_n w_n) = (\lambda_0 - \lambda_0, n) \| \varphi_{0,n} \|^2 + \lambda_0, n (\varphi_{0,n}, w_n) + (\varphi_{0,n}, \eta_n), \]
which can be rewritten to be
\[
\begin{align*}
(\lambda_0 - \lambda_0, n) \| \varphi_{0,n} \|^2 + (\varphi_{0,n}, \eta_n) \\
&= (\varphi_{0,n}, H_n w_n - \lambda_0, n w_n) \\
&= (\varphi_{0,n} - \phi_{0,n} - \alpha_n \phi_{0,n}, H_n w_n - \lambda_0, n w_n) \\
&= (w_n, H_n w_n) - \lambda_0, n \| w_n \|^2 \geq (\lambda_1, n - \lambda_0, n) \| w_n \|^2.
\end{align*}
\] (3.5)

Here, the last inequality comes from \( w_n \in \{ \phi_{0,n} \}^\perp \) and the uniqueness of ground state \( \phi_{0,n} \). Lemma 3.3 yields \( \lambda_1, n - \lambda_0, n \to \lambda_1 - \lambda_0 > 0 \). So combining (3.5) with \( (\varphi_{0,n}, \eta_n) = O(\varepsilon) \) through (3.3), we obtain \( \| w_n \| = O(\varepsilon) \) for big \( n \).

Recall (3.4), it follows \( (1 + \alpha_n)^2 = 1 + O(\varepsilon) \), that is, \( \alpha_n = O(\varepsilon) \) or \( \alpha_n = -2 + O(\varepsilon) \). Assume there is a subsequence \( \alpha_{n_i} = -2 + O(\varepsilon) \), then the nonnegativity of ground state \( \phi_{0,n_i} \) gives
\[ w_{n_i} = \varphi_{0,n_i} - (1 + \alpha_{n_i}) \phi_{0,n_i} \geq \frac{1}{2} \phi_{0,n_i}, \]
which is contradict to \( \| w_n \| = O(\varepsilon) \). So we obtain \( \alpha_n = O(\varepsilon) \) and
\[ \| \varphi_{0,n} \circ \pi_n - \phi_{0,n} \circ \pi_n \|_{\mathcal{Y}}^2 = \| \varphi_{0,n} - \phi_{0,n} \|^2 = \| w_n \|^2 + \alpha_n^2 \| \phi_{0,n} \|^2 = O(\varepsilon), \]
which means \( \| \varphi_{0,n} - \phi_{0,n} \circ \pi_n \|_{\mathcal{Y}} = O(\varepsilon), \) and thus \( \| \phi_{0,n} - \phi_{0,n} \circ \pi_n \|_{\mathcal{Y}} \to 0. \)

Furthermore, the same method in Step 2 of Lemma 2.8 gives
\[ E_\varepsilon[\phi_{0,n} - \phi_{0,n} \circ \pi_n] = \int \| \nabla (\phi_{0,n} - \phi_{0,n} \circ \pi_n) \|^2 \, d\mu \to 0. \]

The proof is completed.

**Proof.** By using the same arguments as in Proposition 2.7, all the conditions in Lemma 3.4 can be derived from original assumptions in Theorem 1.2. Thus we finish the proof by combining Proposition 2.7 and Lemma 3.4. □
4 Spectral gap comparison for diffusion-type operators

In this section we consider the diffusion type operators

\[-L = -L_* + \nabla F \cdot \nabla\]

on the Wiener space \((\mathcal{W}, \mathbb{H}, \mu)\), where \(L_*\) and \(\nabla\) are still the Ornstein–Uhlenbeck operator and the Malliavin derivative operator, respectively, and \(F : \mathcal{W} \to \mathbb{R}\) is some Wiener functional. Our purpose is to compare the spectral gap of \(-L\) with that of the one-dimensional operator 

\[-\tilde{\mathcal{L}} = -\frac{d^2}{dt^2} + (t + \omega(t)) \frac{d}{dt} \]

Note that for the diffusion type operators, 0 is a trivial eigenvalue, hence the spectral gap of \(-L\) (resp. \(-\tilde{\mathcal{L}}\)) coincides with the first non-trivial eigenvalue \(\lambda_1\) (resp. \(\tilde{\lambda}_1\)).

**Remark 4.1.** We note that the spectrum of \(-L = -L_* + \nabla F \cdot \nabla\) coincide with that of the Schrödinger type operator \(-\mathcal{L} = -L_* + V\) where the functional \(V = -\frac{1}{2} L_* F + \frac{1}{4} \|\nabla F\|_{L^2}^2\). Indeed, if \(\phi\) is an eigenfunction of \(-L\) corresponding to the eigenvalue \(\lambda\), then it is straightforward to check that \(\psi := e^{-F/2} \phi\) is an eigenfunction of \(-\mathcal{L}\) corresponding to \(\lambda\). Therefore, to compare the spectral gap of \(-L\) with the one-dimensional operator \(-\tilde{\mathcal{L}}\), we can apply the spectral gap comparison theorem for the Schrödinger operator \(-\mathcal{L}\) by directly imposing conditions of modulus of convexity on the functional \(V\). However, such conditions involve the third order derivatives of \(F\) which are difficult to check.

We divide the proof into three steps: firstly we prove the comparison theorem in the finite dimensional case and the operator \(-L\) is restricted to a bounded convex domain \(\Omega \subset \mathbb{R}^n\); secondly we show that it holds on the whole \(\mathbb{R}^n\) by taking the limit \(\Omega \to \mathbb{R}^n\); finally we use the finite dimensional approximation to prove the spectral gap comparison theorem on the Wiener space.

4.1 The case of a bounded convex domain

In this subsection we consider the diffusion operator \(L = \Delta - \nabla U \cdot \nabla\) on the bounded convex domain \(\Omega \subset \mathbb{R}^n\) with smooth boundary, in which \(U \in W^{1,p}_\text{loc}(\Omega_1)\) and \(\Omega_1 := \{x \in \mathbb{R}^n : \text{dist}(x, \Omega) < 1\}\) is the 1-neighborhood of \(\Omega\). Here we assume \(U\) is defined on \(\Omega_1\) in order to carry out the regularization procedure by convolution. Let \(\lambda_1 = \lambda_1(\Omega)\) (resp. \(\tilde{\lambda}_1 = \lambda_1(D)\)) be the first nontrivial eigenvalue of \(-L\) (resp. \(-\tilde{\mathcal{L}} = -\frac{d^2}{dt^2} + u(t) \frac{d}{dt}\)) on \(\Omega\) (resp. the interval \([-D/2, D/2]\), \(D\) being the diameter of \(\Omega\)) with the Neumann boundary condition. We shall prove \(\lambda_1(\Omega) \geq \lambda_1(D)\) provided that

\[
\langle \nabla U(x) - \nabla U(y) \rangle \cdot \frac{x - y}{|x - y|} \geq 2u' \left( \frac{|x - y|}{2} \right)
\]  

(4.1)

for a.e. \(x, y \in \Omega_1\).

First of all we assume that \(U \in C^1(\Omega) \cap C(\overline{\Omega})\) and prove an analogue of [1, Theorem 1.3]. To this end we need

**Proposition 4.2.** Let \(\lambda_1(\Omega')\) be the first non-trivial eigenvalue of \(-L\) on the domain \(\Omega' \subset \Omega\) with Neumann boundary condition. Then \(\lim_{\Omega' \uparrow \Omega} \lambda_1(\Omega') \leq \lambda_1(\Omega)\).

**Proof.** We shall make use of the variational formulae

\[
\lambda_1(\Omega) = \inf \left\{ \int_{\Omega} |\nabla f|^2 e^{-U} \, dx : f \in C^\infty(\Omega), \int_{\Omega} f e^{-U} \, dx = 0 \text{ and } \int_{\Omega} f^2 e^{-U} \, dx = 1 \right\}
\]  

(4.2)

\[
\lambda_1(\Omega) = \inf \left\{ \int_{\Omega} |\nabla f|^2 e^{-U} \, dx : f \in C^\infty(\Omega), \int_{\Omega} f e^{-U} \, dx = 0 \text{ and } \int_{\Omega} f^2 e^{-U} \, dx = 1 \right\},
\]  

(4.3)
where \( f \in C^\infty_N(\Omega) \) means that \( f \in C^\infty(\Omega) \) and \( f \) satisfies the Neumann boundary condition. We fix any \( \varepsilon > 0 \). By (4.3), we can find \( f \in C^\infty_N(\Omega) \) verifying \( \int_{\Omega} f e^{-U} \, dx = 0 \) and \( \int_{\Omega} f^2 e^{-U} \, dx = 1 \), such that
\[
\lambda_1(\Omega) + \varepsilon > \int_{\Omega} |\nabla f|^2 e^{-U} \, dx. \tag{4.4}
\]
Note that \( f \) and \( |\nabla f| \) are bounded on the closure \( \overline{\Omega} \). For a given subset \( \Omega' \subset \Omega \), we define \( m_{\Omega'} = \int_{\Omega'} f e^{-U} \, dx \) and \( \sigma_{\Omega'} = \left[ \int_{\Omega'} (f - m_{\Omega'})^2 e^{-U} \, dx \right]^{1/2} \). By the choice of \( f \), it is clear that as \( \Omega' \) increases to \( \Omega \), \( m_{\Omega'} \rightarrow 0 \) and
\[
\sigma_{\Omega'}^2 = \int_{\Omega'} f^2 e^{-U} \, dx - 2 m_{\Omega'}^2 + m_{\Omega'}^2 \int_{\Omega'} e^{-U} \, dx \rightarrow 1.
\]
Set \( f_{\Omega'}(x) = [f(x) - m_{\Omega'}]/\sigma_{\Omega'} \) for \( x \in \Omega' \); then \( f_{\Omega'} \in C^\infty(\Omega') \), \( \int_{\Omega'} f_{\Omega'} e^{-U} \, dx = 0 \) and \( \int_{\Omega'} f_{\Omega'}^2 e^{-U} \, dx = 1 \). Therefore by the variational formula (4.2),
\[
\lambda_1(\Omega') \leq \int_{\Omega'} |\nabla f_{\Omega'}|^2 e^{-U} \, dx \leq \frac{1}{\sigma_{\Omega'}^2} \int_{\Omega'} |\nabla f|^2 e^{-U} \, dx < \frac{\lambda_1(\Omega) + \varepsilon}{\sigma_{\Omega'}^2},
\]
where the last inequality follows from (4.4). As a result,
\[
\lim_{\Omega' \uparrow \Omega} \lambda_1(\Omega') \leq \lambda_1(\Omega) + \varepsilon.
\]
The result follows from the arbitrariness of \( \varepsilon \). \( \square \)

**Proposition 4.3.** Suppose that \( U \in C^1(\Omega) \cap C(\overline{\Omega}) \) and for all \( x, y \in \Omega \), it holds
\[
(\nabla U(x) - \nabla U(y)) \cdot \frac{x - y}{|x - y|} \geq 2 u' \left( \frac{|x - y|}{2} \right). \tag{4.5}
\]
Assume \( u \in C^1([-D/2, D/2]) \) is an even function. Then \( \lambda_1(\Omega) \geq \bar{\lambda}_1(D) \).

**Proof.** Fix a convex set \( \Omega' \subset \Omega \) with smooth boundary \( \partial \Omega' \) and \( D' := \text{diam}(\Omega') < D \). Let \( f_{\Omega'} \) be the eigenfunction associated to \( \lambda_1(\Omega') \) of \( -\Delta \) on \( \Omega' \) with Neumann boundary condition, and \( f(t, \cdot) := e^{-t\lambda_1(\Omega')} f_{\Omega'} \). Then \( f \) satisfies
\[
\begin{aligned}
\frac{\partial}{\partial t} f &= \Delta f - \nabla U \cdot \nabla f \quad \text{in } \Omega' \times \mathbb{R}_+; \\
D_v f &= 0 \quad \text{in } \partial \Omega' \times \mathbb{R}_+.
\end{aligned} \tag{4.6}
\]
Because of (4.5), we know that \( X := -\nabla U \) has modulus of contraction \( u' \), hence condition 1 in [1, Theorem 2.1] is satisfied.

Next let \( \phi \) be the eigenfunction associated to \( \bar{\lambda}_1(D) \) of \( -\bar{\Delta} \) on the interval \([-D/2, D/2]\) with the Neumann boundary condition, and \( \varphi(t, \cdot) := C e^{-t\bar{\lambda}_1(D)} \phi \) where \( C > 0 \) is some constant. We may assume \( \phi \) is odd on \([-D/2, D/2]\) and \( \varphi'(s) > 0 \) for all \( 0 \leq s < D/2 \) (see the Appendix for a proof). Recall that \( D' = \text{diam}(\Omega') < D \), thus \( \varphi'(s) > 0 \) for all \( s \in [0, D']/2 \). Since \( f_{\Omega'} \) is Lipschitz continuous, we can take \( C \) big enough such that \( f(0, \cdot) = f_{\Omega'} \) has modulus of continuity \( \varphi(0, \cdot) = C \phi \). So the second condition in [1, Theorem 2.1] is also verified. Finally, all the assumptions in condition 3 of [1, Theorem 2.1] are satisfied on the smaller interval \([0, D']/2\]. Therefore the conclusion of [1, Theorem 2.1] tells us that \( \varphi(t, \cdot) \) is a modulus of continuity of \( f(t, \cdot) \) for all \( t \geq 0 \). Now we deduce easily that \( \lambda_1(\Omega') \geq \bar{\lambda}_1(D) \). Letting \( \Omega' \) increase to \( \Omega \) and applying Proposition 4.2, we finish the proof. \( \square \)
Now we are at the position to extend the above results to the case where the function $U$ is not smooth, that is, $U \in W^{1,p}(\Omega_1)$. Recall that $\Omega_1$ is the 1-neighborhood of $\Omega$. The reason for requiring $U$ is defined on a bigger domain is that we shall use the convolution to regularize $U$.

Take $\rho \in C_\infty^\infty(\mathbb{R}^n, \mathbb{R}_+)$ such that $\text{supp}(\rho) \subset B_1 := \{x \in \mathbb{R}^n : |x| \leq 1\}$ and $\int_{\mathbb{R}^n} \rho(x) \, dx = 1$.

For any $\theta > 0$, define $\rho_\theta(x) = \theta^{-n} \rho(x/\theta)$, $x \in \mathbb{R}^n$. Then $\{\rho_\theta\}_{\theta > 0}$ is a standard family of convolution kernels. For any $\theta \in (0,1)$, set $U_\theta = U * \rho_\theta$; then $U_\theta \in C^\infty(\Omega_1 - \theta)$.

**Lemma 4.4.** Assume that (4.1) holds. Then for any $\theta \in (0,1)$ and all $x \neq y \in \Omega$, one has
\[
(\nabla U_\theta(x) - \nabla U_\theta(y)) \cdot \frac{x - y}{|x - y|} \geq 2u^\prime \left( \frac{|x - y|}{2} \right).
\]

**Proof.** Fix $\theta \in (0,1)$. We have for given $x \neq y \in \Omega$,
\[
\nabla U_\theta(x) - \nabla U_\theta(y) = \int_{B_\theta} \rho_\theta(z) [\nabla U(x - z) - \nabla U(y - z)] \, dz,
\]
where $B_\theta = \{z \in \mathbb{R}^n : |z| \leq \theta\}$. Thus
\[
(\nabla U_\theta(x) - \nabla U_\theta(y)) \cdot \frac{x - y}{|x - y|} = \int_{B_\theta} \rho_\theta(z) [\nabla U(x - z) - \nabla U(y - z)] \cdot \frac{x - y}{|x - y|} \, dz.
\]

For a.e. $z \in B_\theta$, $x - z, y - z \in \Omega_\theta$, hence by (4.1),
\[
[\nabla U(x - z) - \nabla U(y - z)] \cdot \frac{x - y}{|x - y|} = \frac{(x - z) - (y - z)}{|(x - z) - (y - z)|} \cdot \frac{|(x - z) - (y - z)|}{2} \geq \frac{2u^\prime \left( \frac{|x - y|}{2} \right)}{2} = 2u^\prime \left( \frac{|x - y|}{2} \right).
\]

Now the desired result follows since $\int_{B_\theta} \rho_\theta(z) \, dz = 1$. \qed

From Proposition 4.3 and Lemma 4.4, we get

**Corollary 4.5.** For any $\theta > 0$, denote by $\lambda_1(\Omega, \theta)$ the first non-trivial eigenvalue of the operator $-L_\theta = -\Delta + \nabla U_\theta \cdot \nabla$ on the domain $\Omega$ with the Neumann boundary condition. Assume $u \in C^1([-D/2, D/2])$ is even. If (4.1) holds, then one has
\[
\lambda_1(\Omega, \theta) \geq \tilde{\lambda}_1(D).
\]

In the rest of this subsection, we show that $\lim_{\theta \to 0} \lambda_1(\Omega, \theta) \leq \lambda_1(\Omega)$ which will lead to the main result of this subsection. First we give a technical preparation.

**Lemma 4.6.** Assume that (4.1) holds and $u \in C^1([-D/2, D/2])$. Then the functions $U$ and $U_\theta$ are bounded from below on $\Omega$. More precisely, there is $M > 0$ such that for a.e. $x \in \Omega$, it holds $U(x) \wedge U_\theta(x) \geq -M$.

**Proof.** Since $U \in W^{1,p}(\Omega)$, there exists $M_1 > 0$ such that the set $K := \{x \in \Omega : |U(x)| \vee |\nabla U(x)| \leq M_1 - 1\}$ has positive Lebesgue measure. It is well known that $\lim_{\theta \to 0} \|U_\theta - U\|_{W^{1,p}(\Omega)} = 0$. Thus we can find a negligible set $N \subset \Omega$ and a subsequence $\{\theta_k\}$ such that, for all $x \in \Omega \setminus N$, it holds $U_{\theta_k}(x) \to U(x)$ and $\nabla U_{\theta_k}(x) \to \nabla U(x)$ as $k$ tends to $\infty$. To simplify the notations, we assume the convergences hold as $\theta \to 0$. The set $\Omega \setminus N$ is not empty, and we fix $x_0 \in \Omega \setminus N$. There exists $\theta_0 \in (0,1)$ such that for all $0 < \theta \leq \theta_0$, we have $|U_\theta(x_0)| \vee |\nabla U_\theta(x_0)| \leq M_1$. 

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By the fundamental theorem of calculus, we have for any \( x \in \Omega, \)

\[
U_\theta(x) - U_\theta(x_0) = \int_0^1 \langle \nabla U_\theta(x_0 + t(x - x_0)), x - x_0 \rangle \, dt \\
= \int_0^1 \langle \nabla U_\theta(x_0 + t(x - x_0)) - \nabla U_\theta(x_0), x - x_0 \rangle \, dt + \langle \nabla U_\theta(x_0), x - x_0 \rangle.
\]

Lemma 4.4 gives us

\[
U_\theta(x) - U_\theta(x_0) \geq |x - x_0| \int_0^1 2u'(t|x - x_0|) \, dt - M_1|x - x_0| \\
\geq 4 \int_0^{(x-x_0)/2} u'(s) \, ds - M_1 D.
\]

Noticing that \( |U_\theta(x_0)| \leq M_1 \) whenever \( \theta \in (0, \theta_0], \) and \( u' \) is bounded on \([0, D/2], \) hence we can find \( M > 0 \) such that \( U_\theta(x) \geq -M \) for all \( x \in \Omega. \) Finally for all \( x \in \Omega \setminus N, \) \( U_\theta(x) \) converges to \( U(x) \) as \( \theta \to 0 \) which implies that \( U(x) \geq -M. \)

**Proposition 4.7.** Assume that \( (4.1) \) holds and \( u' \in L^1([0, D/2], dt). \) Then one has

\[
\lim_{\theta \to 0} \lambda_1(\Omega, \theta) \leq \lambda_1(\Omega).
\]

**Proof.** Fix any \( \varepsilon > 0. \) By the variational formula \( (4.3), \) there is \( f \in C^\infty_N(\Omega) \) satisfying \( \int_\Omega f e^{-U} \, dx = 0 \) and \( \int_\Omega f^2 e^{-U} \, dx = 1, \) such that

\[
\lambda_1(\Omega) + \varepsilon > \int_\Omega |\nabla f|^2 e^{-U} \, dx. \tag{4.7}
\]

Note that \( f \) and \( |\nabla f| \) are bounded on \( \bar{\Omega}; \) furthermore, \( e^{-U_\theta} \) is bounded from above when \( \theta \) is sufficiently small by Lemma 4.6. Then as \( \theta \) tends to 0, the dominated convergence theorem gives us \( m_\theta := \int_\Omega f e^{-U_\theta} \, dx \to 0 \) and

\[
s_\theta^2 := \int_\Omega (f - m_\theta)^2 e^{-U_\theta} \, dx = \int_\Omega f^2 e^{-U_\theta} \, dx - 2m_\theta^2 + m_\theta^2 \int_\Omega e^{-U_\theta} \, dx \\
\to \int_\Omega f^2 e^{-U} \, dx = 1.
\]

Set \( f_\theta = (f - m_\theta)/s_\theta; \) then \( f_\theta \in C^\infty_N(\Omega), \) \( \int_\Omega f_\theta e^{-U_\theta} \, dx = 0 \) and \( \int_\Omega f_\theta^2 e^{-U_\theta} \, dx = 1. \) Again by the variational formula \( (4.3), \)

\[
\lambda_1(\Omega, \theta) \leq \int_\Omega |\nabla f_\theta|^2 e^{-U_\theta} \, dx = \frac{1}{s_\theta^2} \int_\Omega |\nabla f|^2 e^{-U_\theta} \, dx.
\]

Thus by the dominated convergence theorem and \( (4.7), \)

\[
\lim_{\theta \to 0} \lambda_1(\Omega, \theta) \leq \int_\Omega |\nabla f|^2 e^{-U} \, dx < \lambda_1(\Omega) + \varepsilon.
\]

Letting \( \varepsilon \) decrease to 0, we complete the proof.

Combining Corollary 4.5 and Proposition 4.7, we obtain the main result of this subsection.

**Theorem 4.8.** Assume that \( U \in W^{1,p}(\Omega_1), \) \( (4.1) \) holds and \( u \in C^1([-D/2, D/2]) \) is even. Then we have \( \lambda_1(\Omega) \geq \lambda_1(D). \)
4.2 The comparison theorem for diffusion operators on $\mathbb{R}^n$

In the present subsection we extend Theorem 4.8 to the whole space by letting $\Omega$ grow to $\mathbb{R}^n$. We assume that $U \in W_{1,p}^1(\mathbb{R}^n)$, satisfying $\int_{\mathbb{R}^n} e^{-U} \, dx = 1$ and for a.e. $x \neq y \in \mathbb{R}^n$,

$$\langle \nabla U(x) - \nabla U(y) \rangle \cdot \frac{x - y}{|x - y|} \geq 2u'\left(\frac{|x - y|}{2}\right).$$  \hfill (4.8)

First we recall the variational formula for the first non-trivial eigenvalue $\lambda_1$ of $-\mathcal{L} = -\Delta + \nabla U \cdot \nabla$ on $\mathbb{R}^n$:

$$\lambda_1 = \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^2 e^{-U} \, dx : f \in C_c^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} f e^{-U} \, dx = 0 \text{ and } \int_{\mathbb{R}^n} f^2 e^{-U} \, dx = 1 \right\}$$ \hfill (4.9)

$$= \inf \left\{ \int_{\mathbb{R}^n} |\nabla f|^2 e^{-U} \, dx : f \in C_c^\infty(\mathbb{R}^n), \int_{\mathbb{R}^n} f e^{-U} \, dx = 0 \text{ and } \int_{\mathbb{R}^n} f^2 e^{-U} \, dx = 1 \right\},$$ \hfill (4.10)

where $C_c^\infty(\mathbb{R}^n) = \mathbb{R} + C_c^\infty(\mathbb{R}^n)$ (see [9, page 51] for the second formula). We shall use both formulae according to the situation. We also assume that $u \in C^1(\mathbb{R})$ is even such that $\int_{\mathbb{R}} e^{-u} \, dt = 1$. The first non-trivial eigenvalue $\tilde{\lambda}_1$ of $-\tilde{\mathcal{L}} = -\frac{d^2}{dt^2} + u'(t) \frac{dt}{dt}$ has the same variational formulae as (4.9) and (4.10), replacing $\mathbb{R}^n$ by $\mathbb{R}$.

To begin with, we can easily prove

**Proposition 4.9.** Let $\lambda_1(\Omega)$ (resp. $\tilde{\lambda}_1(D)$) be the first non-trivial eigenvalue of $-\mathcal{L} = -\Delta + \nabla U \cdot \nabla$ (resp. $-\tilde{\mathcal{L}} = -\frac{d^2}{dt^2} + u'(t) \frac{dt}{dt}$) on the bounded convex domain $\Omega$ (resp. the interval $[-D/2, D/2]$) with the Neumann boundary condition. Then $\lim_{\Omega \uparrow \mathbb{R}^n} \lambda_1(\Omega) \leq \lambda_1$ and $\lim_{D \to \infty} \tilde{\lambda}_1(D) \leq \tilde{\lambda}_1$.

**Proof.** Fix any $\varepsilon > 0$. By the variational expression (4.9), there exists $f \in C_c^\infty(\mathbb{R}^n)$ verifying $\int_{\mathbb{R}^n} f e^{-U} \, dx = 0$ and $\int_{\mathbb{R}^n} f^2 e^{-U} \, dx = 1$, such that

$$\lambda_1 + \varepsilon > \int_{\mathbb{R}^n} |\nabla f|^2 e^{-U} \, dx.$$ \hfill (4.11)

When the domain $\Omega$ is big enough, we have $f \in C_c^\infty(\Omega)$, and $\int_{\Omega} f e^{-U} \, dx = 0$, $\int_{\Omega} f^2 e^{-U} \, dx = 1$. Therefore by (4.2), we obtain

$$\lambda_1(\Omega) \leq \int_{\Omega} |\nabla f|^2 e^{-U} \, dx = \int_{\mathbb{R}^n} |\nabla f|^2 e^{-U} \, dx < \lambda_1 + \varepsilon,$$

where the last inequality follows from (4.11). Thus $\lim_{\Omega \uparrow \mathbb{R}^n} \lambda_1(\Omega) \leq \lambda_1$. The proof of the second result is analogous, hence we omit it. \qed

Next we show that in the one dimensional case the stronger result holds.

**Proposition 4.10.** Assume that $u'(s) \to +\infty$ as $s$ tends to $+\infty$. Then $\lim_{D \to \infty} \tilde{\lambda}_1(D) = \tilde{\lambda}_1$.

**Proof.** It suffices to prove that $\lim_{D \to \infty} \tilde{\lambda}_1(D) \geq \tilde{\lambda}_1$. Fix any $\varepsilon \in (0, 1)$. For each $D > 0$, we choose $f_D \in C^\infty(-D/2, D/2)$ satisfying

$$f_D'(\pm D/2) = 0, \quad \int_{-\frac{D}{2}}^{\frac{D}{2}} f_D e^{-u} \, ds = 0 \quad \text{and} \quad \int_{-\frac{D}{2}}^{\frac{D}{2}} f_D^2 e^{-u} \, ds = 1,$$
such that
\[ \int_{-\frac{D}{2}}^{\frac{D}{2}} |f_D'|^2 e^{-u} \, ds < \lambda_1(D) + \varepsilon. \] (4.12)

Taking into account the result in Proposition 4.9, we may assume that for $D$ big enough,
\[ \int_{-\frac{D}{2}}^{\frac{D}{2}} |f_D'|^2 e^{-u} \, ds < \lambda_1 + 1. \] (4.13)

We extend $f_D$ to a function $\tilde{f}_D \in C_0^\infty(\mathbb{R})$ as follows: $\tilde{f}_D(s) = f_D(s)$ for all $s \in [-D/2, D/2]$, and $|\tilde{f}_D(s)| \leq \sqrt{\varepsilon}$ for all $|s| > D/2$, $\sup_{s > D/2} |f_D(s)| \leq |f_D(D/2)| + 1$, $\sup_{s < -D/2} |\tilde{f}_D(s)| \leq |f_D(-D/2)| + 1$.

First of all we prove an estimate of $|f_D(\pm D/2)|$ in terms of $\lambda_1$. Define $Z_D = \int_{-D/2}^{D/2} e^{-u(s)} \, ds$. Then $Z_D \uparrow 1$ as $D$ tends to $\infty$. By the choice of $f_D$, we have
\[
Z_D f_D(D/2) = Z_D f_D(D/2) - \int_{-\frac{D}{2}}^{\frac{D}{2}} f_D e^{-u} \, ds = \int_{-\frac{D}{2}}^{\frac{D}{2}} \left[ f_D(D/2) - f_D(s) \right] e^{-u(s)} \, ds
\]
\[
= \int_{-\frac{D}{2}}^{\frac{D}{2}} \left[ \int_{s}^{D/2} f_D(t) \, dt \right] e^{-u(s)} \, ds = \int_{-\frac{D}{2}}^{\frac{D}{2}} f_D(t) \left[ \int_{-\frac{D}{2}}^{t} e^{-u(s)} \, ds \right] dt,
\]
where in the last step we changed the order of integration. Noticing that $\int_{\mathbb{R}} e^{-u(s)} \, ds = 1$, we have by Cauchy’s inequality and (4.13) that
\[
|Z_D f_D(D/2)| \leq \int_{-\frac{D}{2}}^{\frac{D}{2}} |f_D(t)| \, dt = \int_{-\frac{D}{2}}^{\frac{D}{2}} |f_D(t)| e^{-u(t)/2} e^{u(t)/2} \, dt
\]
\[
\leq \left[ \int_{-\frac{D}{2}}^{\frac{D}{2}} |f_D(t)|^2 e^{-u(t)} \, dt \right]^{1/2} \left[ \int_{-\frac{D}{2}}^{\frac{D}{2}} e^{u(t)} \, dt \right]^{1/2}
\]
\[
< (\lambda_1 + 1)^{1/2} \left[ \int_{-\frac{D}{2}}^{\frac{D}{2}} e^{u(t)} \, dt \right]^{1/2}. \] (4.14)

The same estimate holds for $|Z_D f_D(-D/2)|$.

Now we have
\[
\sigma_D^2 := \int_{\mathbb{R}} |\tilde{f}_D(t)|^2 e^{-u(t)} \, dt = \left( \int_{-\infty}^{-\frac{D}{2}} + \int_{\frac{D}{2}}^{\infty} \right) |\tilde{f}_D(t)|^2 e^{-u(t)} \, dt,
\]
therefore by (4.14) and the definition of $\tilde{f}_D$,
\[
|\sigma_D^2 - 1| \leq (|f_D(-D/2)| + 1)^2 \int_{-\infty}^{-\frac{D}{2}} e^{-u(t)} \, dt + (|f_D(D/2)| + 1)^2 \int_{\frac{D}{2}}^{\infty} e^{-u(t)} \, dt
\]
\[
\leq 4 \left[ 1 + 2(\lambda_1 + 1) Z_D^{-1} \int_{0}^{\frac{D}{2}} e^{u(t)} \, dt \right] \int_{\frac{D}{2}}^{\infty} e^{-u(t)} \, dt
\](4.15)
since $u$ is even. We need the following simple result.

**Lemma 4.11.** Under the assumption of Proposition 4.10, we have
\[
\lim_{s \to \infty} \left( \int_{0}^{s} e^{u(t)} \, dt \right) \int_{s}^{\infty} e^{-u(t)} \, dt = 0.
\]
Proof. Since \( \lim_{s \to +\infty} u'(s) = +\infty \), it holds \( \lim_{s \to +\infty} u(s) = +\infty \). By L'Hôpital's rule,
\[
\lim_{s \to \infty} \frac{\int_0^s e^{u(t)} \, dt}{e^{u(s)}} = \lim_{s \to \infty} \frac{e^{u(s)}}{u'(s)} = \lim_{s \to \infty} \frac{1}{u'(s)} = 0. 
\] (4.16)
Next for \( s \) big enough,
\[
e^{u(s)} \int_s^\infty e^{-u(t)} \, dt \leq e^{u(s)} \int_s^\infty e^{-u(t)} u'(t) \, dt = 1. 
\] (4.17)
Combining (4.16) with (4.17), we obtain the desired result. \( \square \)

Now we continue the proof of Proposition 4.10. By the above lemma and (4.15), we get
\[
\lim_{D \to \infty} \tilde{\sigma}_D^2 = 1. 
\] (4.18)
Next
\[
m_D := \int_\mathbb{R} \tilde{f}_D(t) e^{-u(t)} \, dt = \left( \int_{-\infty}^{-\frac{D}{2}} + \int_{-\frac{D}{2}}^{\frac{D}{2}} + \int_{\frac{D}{2}}^{\infty} \right) \tilde{f}_D(t) e^{-u(t)} \, dt, 
\]
hence by the choice of \( \bar{\lambda} \), (4.14) and Lemma 4.11,
\[
|m_D| \leq (|f_D(-D/2)| + 1) \int_{-\infty}^{-\frac{D}{2}} e^{-u(t)} \, dt + (|f_D(D/2)| + 1) \int_{\frac{D}{2}}^{\infty} e^{-u(t)} \, dt
\]
\[
\leq 2 \left[ \frac{1}{Z_D} \left( 2\bar{\lambda} + 1 \right) \int_0^{\frac{D}{2}} e^{u(t)} \, dt \right]^{\frac{1}{2}} + 1 \int_{\frac{D}{2}}^{\infty} e^{-u(t)} \, dt \to 0 
\] (4.19)
as \( D \) goes to \( \infty \).

Set
\[
\sigma_D^2 := \int_\mathbb{R} \left[ \tilde{f}_D(t) - m_D \right]^2 e^{-u(t)} \, dt = \tilde{\sigma}_D^2 - m_D^2 \to 1 \quad \text{as } D \to \infty, 
\]
thanks to (4.18) and (4.19). Finally define \( \tilde{f}_D(t) = (\tilde{f}_D(t) - m_D)/\sigma_D \) for all \( t \in \mathbb{R} \). Then \( \tilde{f}_D \) is constant outside a compact set, thus \( \tilde{f}_D \in C_\infty^\infty(\mathbb{R}) \). Moreover, \( \int_\mathbb{R} \tilde{f}_D(t) e^{-u(t)} \, dt = 0 \) and \( \int_\mathbb{R} [\tilde{f}_D(t)]^2 e^{-u(t)} \, dt = 1 \). Therefore by the variational formula (4.10), the choice of \( \tilde{f}_D \) and (4.12), we have
\[
\bar{\lambda}_1 \leq \int_\mathbb{R} |\tilde{f}_D(t)|^2 e^{-u(t)} \, dt = \frac{1}{\sigma_D^2} \int_\mathbb{R} [\tilde{f}_D(t)]^2 e^{-u(t)} \, dt
\]
\[
\leq \frac{1}{\sigma_D^2} \left[ \varepsilon + \lambda_1(D) + \varepsilon + \varepsilon \right]. 
\]
As a result,
\[
\bar{\lambda}_1 \leq \lim_{D \to \infty} \bar{\lambda}_1(D) + 3\varepsilon. 
\]
We finish the proof since \( \varepsilon > 0 \) is arbitrary. \( \square \)

We now state the main result of this subsection.

**Theorem 4.12.** Suppose that \( U \in W^{1,p}_{loc}(\mathbb{R}^n) \), satisfying \( \int_{\mathbb{R}^n} e^{-U} \, dx = 1 \) and for a.e. \( x \neq y \in \mathbb{R}^n \),
\[
(\nabla U(x) - \nabla U(y)) \cdot \frac{x - y}{|x - y|} \geq 2u'(\frac{|x - y|}{2}). 
\] (4.20)
Assume also that \( u \in C^1(\mathbb{R}) \) is even, \( \int_{\mathbb{R}} e^{-u} \, dt = 1 \) and \( \lim_{t \to \infty} u'(t) = \infty \). Then we have \( \lambda_1 \geq \bar{\lambda}_1 \).
Proof. Applying Theorem 4.8 on the ball \( B_{D/2} := \{ x \in \mathbb{R}^n : |x| \leq D/2 \} \), we get

\[
\lambda_1(B_{D/2}) \geq \tilde{\lambda}_1(D).
\]

Letting \( D \to \infty \), Propositions 4.9 and 4.10 lead to the result. \( \Box \)

4.3 The comparison theorem for diffusion type operators on the Wiener space

We adopt the notations in Section 3. Suppose we are given a Wiener functional \( F \in \mathbb{D}^1_{\alpha}(\mathbb{W}) \) satisfying \( \int_{\mathbb{W}} e^{-F} \, d\mu = 1 \). We shall compare the first non-trivial eigenvalue \( \lambda_1 \) of the diffusion type operator \(-\mathcal{L} = -\mathcal{L}_s + \nabla F \cdot \nabla \) with \( \tilde{\lambda}_1 \) of the one dimensional operator \(-\tilde{\mathcal{L}} = -\frac{d^2}{dt^2} + (t + \omega'(t)) \frac{1}{1!} \). The two functions \( F \) and \( \omega \) are related by the following inequality: for all \( h \in \mathbb{H} \) and \( \mu\)-a.e. \( w \in \mathbb{W} \),

\[
\left\langle \nabla F(w + h) - \nabla F(w), \frac{h}{\|h\|_{\mathbb{H}}} \right\rangle_{\mathbb{H}} \geq \omega'(\frac{\|h\|_{\mathbb{H}}}{2}). \tag{4.21}
\]

Recall that \( \lambda_1 \) has the variational formula

\[
\lambda_1 = \inf \left\{ \int_{\mathbb{W}} |\nabla \varphi|^2 e^{-F} \, d\mu : \varphi \in \text{Cylin}(\mathbb{W}), \int_{\mathbb{W}} \varphi e^{-F} \, d\mu = 0 \text{ and } \int_{\mathbb{W}} \varphi^2 e^{-F} \, d\mu = 1 \right\}, \tag{4.22}
\]

where \( \text{Cylin}(\mathbb{W}) \) is the collection of cylindrical functionals on \( \mathbb{W} \).

The main result of this subsection is

**Theorem 4.13** (Gap comparison on the Wiener space). Assume that \( F \in \mathbb{D}^1_{\alpha}(\mathbb{W}, \mathbb{R}) \) satisfies \( \int_{\mathbb{W}} e^{-F} \, d\mu = 1 \) and (4.21). Suppose also that \( \omega \in C^1(\mathbb{R}) \) is even, satisfying \( \int_{\mathbb{R}} e^{-\omega} \, d\gamma_1 = 1 \) and \( \lim_{t \to \infty} (\omega'(t) + t) = +\infty \). Then we have \( \lambda_1 \geq \tilde{\lambda}_1 \).

In the remainder of the current subsection, we prove the above theorem. For any \( n \geq 1 \), let \( F_n \in \mathbb{D}^1(\mathbb{X}_n) \) be defined by the identity \( E^{X_n} F = F_n \circ \pi_n \). First we have

**Proposition 4.14.** Under the condition (4.21), it holds

\[
\left\langle \nabla_n F_n(x) - \nabla_n F_n(x'), \frac{x - x'}{|x - x'|} \right\rangle_{\mathbb{X}_n} \geq \omega'(\frac{|x - x'|}{2}) \text{ for a.e. } x, x' \in \mathbb{X}_n.
\]

Proof. The proof is the same as Lemma 3.1. \( \Box \)

Now we present a technical result.

**Lemma 4.15.** Let \( \psi \in \text{Cylin}(\mathbb{W}) \). If \( \int_{\mathbb{W}} |\psi| e^{-F} \, d\mu < +\infty \), then

\[
\lim_{n \to \infty} \int_{\mathbb{X}_n} \psi e^{-F_n} \, d\gamma_n = \int_{\mathbb{W}} \psi e^{-F} \, d\mu.
\]

Proof. Since \( \psi \) is a cylindrical functional, we can regard it as a function in the Schwartz space \( \mathcal{S}(\mathbb{X}_n) \) for every \( n \) big enough. We have

\[
\int_{\mathbb{X}_n} \psi e^{-F_n} \, d\gamma_n = \int_{\mathbb{W}} \psi e^{-F_n \circ \pi_n} \, d\mu. \tag{4.23}
\]

By Jensen’s inequality,

\[
|\psi| e^{-F_n \circ \pi_n} = |\psi| e^{-E^{X_n} F} \leq |\psi| E^{X_n} (e^{-F}) = E^{X_n} (|\psi| e^{-F}),
\]

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where in the last equality we used the property of conditional expectation. Since $|\psi|e^{-F} \in L^1(\mu)$, the family $\{E^{X_n}(\psi|e^{-F}) : n \geq 1\}$ is uniformly integrable, which implies that the family $\{|\psi|e^{-F_n \circ \pi_n} : n \geq 1\}$ is also uniformly integrable. Up to a subsequence, $F_n \circ \pi_n$ converge to $F$ almost surely, hence

$$\lim_{n \to \infty} \int_W \psi e^{-F_n \circ \pi_n} \, d\mu = \int_W \psi e^{-F} \, d\mu.$$  

The proof is finished by combining this result with (4.23). \hfill\qed

Taking $\psi \equiv 1$ in the above lemma, we see that $Z_n := \int_{X_n} e^{-F_n} \, d\gamma_n \to 1$ as $n$ goes to infinity. Denote by $\lambda_1(n)$ the first non-trivial eigenvalue of

$$-\mathcal{L}_n := -\Delta_n + (x + \nabla_n F_n) \cdot \nabla_n,$$

where $\Delta_n$ and $\nabla_n$ are respectively the Laplace operator and the gradient operator on $X_n$ and $x \in X_n$. It has the following expression

$$\lambda_1(n) = \inf \left\{ \frac{1}{Z_n} \int_{X_n} |\nabla_n \varphi|^2 e^{-F_n} \, d\gamma_n : \varphi \in \mathcal{S}(X_n), \int_{X_n} \varphi e^{-F_n} \, d\gamma_n = 0 \right\},$$

where $\mathcal{S}(X_n)$ is the space of Schwartz functions on $X_n$.

**Proposition 4.16.** Assume the conditions in Theorem 4.13. Then for every $n \geq 1$, it holds $\lambda_1(n) \geq \lambda_1$.

**Proof.** We check that the conditions in Theorem 4.12 are satisfied by $U_n(x) = F_n(x) + \frac{1}{2}|x|^2 + \frac{1}{2} \log(2\pi) + \log Z_n (x \in X_n)$ and $u(t) = \omega(t) + \frac{t^2}{2} + \frac{1}{2} \log(2\pi) (t \in \mathbb{R})$.

First it is clear that $U_n \in W^{1,p}_{\text{loc}}(X_n)$ since $F_n \in \mathbb{D}^p(X_n, \gamma_n)$. By the definition of $Z_n$, we have $\int_{X_n} e^{-U_n(x)} \, dx = 1$. From Proposition 4.14, we conclude that $U_n$ has modulus of convexity $u$, that is, (4.8) is verified. By the assumptions on $\omega$, one immediately checks the conditions on $u$. Hence Proposition 4.14 leads to the desired result. \hfill\qed

Now we are ready to prove the main result of this subsection.

**Proof of Theorem 4.13.** It is enough to show that $\lim_{n \to \infty} \lambda_1(n) \leq \lambda_1$. By the variational formula (4.22), for any $\varepsilon > 0$, there is $\varphi \in \text{Cylin}(\mathcal{W})$ verifying $\int_{\mathcal{W}} \varphi e^{-F} \, d\mu = 0$ and $\int_{\mathcal{W}} \varphi^2 e^{-F} \, d\mu = 1$, such that

$$\int_{\mathcal{W}} |\nabla \varphi|^2 e^{-F} \, d\mu < \lambda_1 + \varepsilon.$$  

(4.25)

For $n$ big enough, $\varphi$ can be seen as a Schwartz test function on $X_n$. Set $m_n := \int_{X_n} \varphi e^{-F_n} \, d\gamma_n = \int_{\mathcal{W}} \varphi e^{-F_n \circ \pi_n} \, d\mu$. Since $|\varphi|e^{-F} \in L^1(\mu)$, Lemma 4.15 tells us that $m_n \to 0$ as $n$ tends to $\infty$. Set $\sigma_n^2 = \frac{1}{Z_n} \int_{X_n} (\varphi - m_n)^2 e^{-F_n} \, d\gamma_n$; one has

$$\sigma_n^2 = \frac{1}{Z_n} \int_{X_n} \varphi^2 e^{-F_n} \, d\gamma_n - \frac{2m_n^2}{Z_n} + m_n^2.$$  

As $\lim_{n \to \infty} Z_n = 1$, the last two terms vanish as $n$ goes to $\infty$. Again by Lemma 4.15,

$$\lim_{n \to \infty} \int_{X_n} \varphi^2 e^{-F_n} \, d\gamma_n = \int_{\mathcal{W}} \varphi^2 e^{-F} \, d\mu = 1.$$
Therefore we have \( \lim_{n \to \infty} \sigma_n^2 = 1 \). Now set \( \varphi_n = (\varphi - m_n)/\sigma_n \); then \( \varphi_n \) belongs to the Schwartz space \( \mathcal{S}(X_n) \), \( \int_{X_n} \varphi_n e^{-F_n} \, d\gamma_n = 0 \) and \( \frac{2}{Z_n} \int_{X_n} \varphi_n^2 e^{-F_n} \, d\gamma_n = 1 \). Therefore by the variational formula (4.24) we obtain for \( n \) big enough,

\[
\lambda_1(n) \leq \frac{1}{Z_n} \int_{X_n} |\nabla \varphi_n|^2 e^{-F_n} \, d\gamma_n = \frac{1}{Z_n \sigma_n^2} \int_{X_n} |\nabla \varphi|^2 e^{-F_n} \, d\gamma_n.
\]

Once again we apply Lemma 4.15 to get

\[
\lim_{n \to \infty} \int_{X_n} |\nabla \varphi|^2 e^{-F_n} \, d\gamma_n = \int_\mathcal{W} |\nabla \varphi|^2 e^{-F} \, d\mu.
\]

Notice that \( Z_n \to 1 \) and \( \sigma_n^2 \to 1 \) as \( n \) increases to \( \infty \). Hence by (4.25), we arrive at

\[
\lim_{n \to \infty} \lambda_1(n) \leq \int_\mathcal{W} |\nabla \varphi|^2 e^{-F} \, d\mu < \lambda_1 + \varepsilon.
\]

The proof is finished.

To conclude this section, we give a simple example to compare the condition (4.21) on the modulus of convexity with the exponential integrability discussed in [4, Example 7.3].

**Example 4.17.** We confine ourselves to the one dimensional case, i.e. \( \mathcal{W} = \mathbb{H} = \mathbb{R} \) and \( \mu = \gamma_1 \) is the Gaussian distribution. Let \( F(x) = \frac{x^2}{2} + Z_0 \) where \( Z_0 \in \mathbb{R} \) is a normalizing constant such that \( \int_\mathbb{R} e^{-F} \, d\gamma_1 = 1 \). Clearly, \( F \) satisfies the conditions of Theorem 4.13 and it has a modulus of convexity, but \( F'(x) = x \) does not satisfy the exponential integrability condition [4, (7.6)]. In the following, we give an example in which \( F' \) verifies [4, (7.6)], but \( F \) has no modulus of convexity. The basic idea is to construct a function whose second derivative is not bounded from below.

Let \( l_1 : y = x/4, \ l_2 : y = -x/4 \ (x \geq 0) \) be two radials. We shall define a function \( F \) such that the graph of \( y = F'(x) \) oscillates between these two radials. More precisely, let

\[
F'(2k) = (-1)^{k-1} 2/k, \quad k \in \mathbb{Z}_+;
\]

and for \( x \in [2k, 2k + 2] \), \( y = F'(x) \) is the line segment linking the two points \( (2k, (-1)^{k-1} 2/k) \) and \( (2k + 2, (-1)^k (k + 1)/2) \). Now the function \( F \) is given by \( F(x) = \int_0^x F'(t) \, dt \) for \( x \geq 0 \) and \( F(x) = F(-x) \) for \( x < 0 \). Since \( |F'(x)| \leq |x|/4 \), it satisfies the exponential integrability [4, (7.6)]. Next for \( h \in (0, 2] \), it is clear that

\[
\frac{F'(4k + 2 + h) - F'(4k + 2)}{h} = \frac{F'(4k + 4) - F'(4k + 2)}{2},
\]

which leads to

\[
F'(4k + 2 + h) - F'(4k + 2) = -(k + 3/4) h.
\]

If \( F \) has \( \omega \) as its modulus of convexity, then

\[
2\omega((h/2) \leq -(k + 3/4) h, \quad \text{for all } k \in \mathbb{Z}_+.
\]

Letting \( k \to \infty \), we see that \( \omega\big|_{(0,1]} \equiv -\infty \) which is absurd.
5 Appendix

In this section we prove the assertion in Proposition 4.3, that is, when \( u \in C^4([-D/2, D/2]) \) is even, the eigenfunction \( \phi_1 \) associated to the first non-trivial eigenvalue \( \tilde{\lambda}_1(D) \) of \( -\tilde{\mathcal{L}} = -\frac{d^2}{dx^2} + u'(t)\frac{d}{dt} \) is an odd function, and changing the sign if necessary, we have \( \phi_1'(t) > 0 \) for all \( 0 < t < D/2 \).

Proof. We shall make use of two well-known results in the Sturm–Liouville theory (cf. [8, Theorems 5.9 and 5.20]):

(i) the eigenvalue \( \lambda_1(D) \) has multiplicity one, in other words, the eigen-space corresponding to \( \lambda_1(D) \) is one-dimensional;

(ii) the eigenfunction \( \phi_1 \) has only one zero in the open interval \((-D/2, D/2)\) and

\[
\int_{-D/2}^{D/2} \phi_1(t)e^{-u(t)} \, dt = 0.
\]

Note that \( \phi_1 \) satisfies the Neumann boundary condition at \( D/2 \), i.e. \( \phi_1(D/2) = 0 \), then it is known that \( \phi_1(D/2) \neq 0 \). Without loss of generality, we assume \( \phi_1(D/2) > 0 \).

To prove the first assertion, define a new function \( \psi(x) = -\phi_1(-x), \forall x \in [-D/2, D/2] \). Then \( \psi'(x) = \phi_1'(-x) \) and \( \psi''(x) = -\phi_1''(-x) \). As \( u \) is even, \( u'(x) = -u'(-x) \). Since \( \psi''(x) = -\phi_1''(x) + u'(x)\phi_1'(x) = \tilde{\lambda}_1(D)\phi_1(x) \),

we have

\[
\psi''(x) = -\phi_1''(x) + \tilde{\lambda}_1(D)\phi_1(x) - u'(x)\phi_1'(x) = -\tilde{\lambda}_1(D)\psi_1(x) + u'(x)\psi'(x).
\]

Therefore \( \psi \) is also an eigenfunction associated to \( \tilde{\lambda}_1(D) \). From (i) we conclude that \( \psi = c\phi_1 \) for some constant \( c \in \mathbb{R}, c \neq 0 \). By (ii), \( \phi_1 \) has exactly one zero, say \( x_0 \), in the interval \((-D/2, D/2)\). Then we deduce from \( -\phi_1(-x_0) = c\phi_1(x_0) = 0 \) that \( x_0 = 0 \). Noticing that \( \phi_1(D/2) > 0 \), \( \phi_1 \) has to be strictly positive on the interval \((0, D/2)\). Thus \( \phi_1'(0) > 0 \), which implies \( \phi_1 \) is negative in a small neighborhood \((-\delta, 0)\). But \( \phi_1 \) has only one zero in the interval \((-D/2, D/2)\), we conclude that \( \phi_1 \) is strictly negative on the interval \([-D/2, 0)\). This also implies that \( c > 0 \). To show that \( c = 1 \), we note that

\[
0 = \int_{-D/2}^{D/2} \phi_1(x)e^{-u(x)} \, dx = \int_{0}^{D/2} \phi_1(x)e^{-u(x)} \, dx + \int_{-D/2}^{0} \phi_1(x)e^{-u(x)} \, dx.
\]

Changing the variable \( t = -x \) in the second integral and using the fact that \( u \) is even, we get

\[
\int_{-D/2}^{0} \phi_1(x)e^{-u(x)} \, dx = \int_{0}^{D/2} \phi_1(-t)e^{-u(t)} \, dt = -c \int_{0}^{D/2} \phi_1(t)e^{-u(t)} \, dt.
\]

Substituting this quantity into (5.2) we obtain \( c = 1 \). Hence \( \phi_1 \) is odd.

It remains to prove that \( \phi_1'(t) > 0 \) when \( 0 < t < D/2 \). Since \( \phi_1(0) = 0 \) and \( \phi_1[0,D/2] > 0 \), we have \( \phi_1'(0) > 0 \). If there exists \( t \in (0, D/2) \) such that \( \phi_1'(t) \leq 0 \), we denote by \( t_1 \) the smallest zero of \( \phi_1' \). According to the equation (5.1), we have \( \phi_1''(t_1) = -\tilde{\lambda}_1(D)\phi_1(t_1) < 0 \). Therefore, we can find \( \delta > 0 \) small enough, such that \( \phi_1'[t_1,t_1+\delta] < 0 \). Noting that \( \phi_1'(D/2) = 0 \), we can define

\[
t_2 = \inf \{ t \in (t_1, D/2) : \phi_1'(t) = 0 \}.
\]

Then \( \phi_1'[t_1,t_2] < 0 \) and \( \phi_1'(t_2) = 0 \). Therefore \( \phi_1''(t_2) \geq 0 \). Letting \( t = t_2 \) in (5.1), we get \( \tilde{\lambda}_1(D)\phi_1(t_2) = -\phi_1''(t_2) \leq 0 \). This contradicts the fact that \( \phi_1 \) is positive on the interval \((0, D/2)\). Therefore \( \phi_1' \) is positive on the interval \([0, D/2)\). \( \square \)
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