CLASSIFICATION OF NONNEGATIVE SOLUTIONS TO A
BI-HARMONIC EQUATION WITH HARTREE TYPE
NONLINEARITY

DAOMIN CAO, WEI DAI

Abstract. In this paper, we are concerned with the following bi-harmonic equation
with Hartree type nonlinearity
\[(P_\gamma) \quad \Delta^2 u = \left( \frac{1}{|x|^8} * |u|^2 \right) u^\gamma, \quad x \in \mathbb{R}^d, \]
where \(0 < \gamma \leq 1\) and \(d \geq 9\). By applying the method of moving planes, we prove
that nonnegative classical solutions \(u\) to \((P_\gamma)\) are radially symmetric about some point
\(x_0 \in \mathbb{R}^d\) and derive the explicit form for \(u\) in the \(\dot H^2\) critical case \(\gamma = 1\). We also prove
the non-existence of nontrivial nonnegative classical solutions in the subcritical cases
\(0 < \gamma < 1\). As a consequence, we also derive the best constants and extremal functions
in the corresponding Hardy-Littlewood-Sobolev inequalities.

Keywords: Bi-harmonic; Nonnegative solutions; Liouville type theorems; Radial symmetry;
Hartree type nonlinearity; Methods of moving planes.

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1. Introduction

In this paper, we consider nonnegative solutions to the following bi-harmonic equation
with Hartree type nonlinearity
\[(P_\gamma) \quad \Delta^2 u = \left( \frac{1}{|x|^8} * |u|^2 \right) u^\gamma, \quad x \in \mathbb{R}^d, \]
where \(0 < \gamma \leq 1\) and \(d \geq 9\). The equation \((P_\gamma)\) is equivalent to the following integral
equation
\[u(x) = \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-4}} \left( \int_{\mathbb{R}^d} \frac{|u(z)|^2}{|y-z|^8} dz \right) u^\gamma(y) dy. \]

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When $\gamma = 1$, equation $(\mathcal{P}_\gamma)$ takes the form of

$$(\mathcal{P}_1) \quad \Delta^2 u = \left( \frac{1}{|x|^8} * |u|^2 \right) u, \quad x \in \mathbb{R}^d,$$

which is $\dot{H}^2$ critical in the sense that both the equation $(\mathcal{P}_1)$ and the $\dot{H}^2$ norm are invariant under the scaling $u_\rho(x) = \rho^{\frac{d+4}{2}} u(\rho x)$.

The solution $u$ to problem $(\mathcal{P}_1)$ is also a ground state or a stationary solution to the following $\dot{H}^2$ critical focusing fourth-order Hartree equation

$$(1.2) \quad i \partial_t u + \Delta^2 u = \left( \frac{1}{|x|^8} * |u|^2 \right) u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,$$

where $d \geq 9$. The Hartree equation with fourth-order dispersion term has many interesting applications in the quantum theory of large systems of non-relativistic bosonic atoms and molecules and the theory of laser propagation in medium (see, e.g. [9, 13]). The bi-harmonic Schrödinger equations and Hartree equations have been quite intensively studied, please refer to [20, 22, 23, 24, 25], in which the ground state solution can be regarded as a crucial criterion or threshold for global well-posedness and scattering in the focusing case. Therefore, the classification of solutions to $(\mathcal{P}_\gamma)$ plays an important and fundamental role in the study of the focusing bi-harmonic Hartree equation (1.2).

In this paper, using the method of moving planes, we study the classification results for nonnegative classical solutions $u \in C^4(\mathbb{R}^d)$ to the equation $(\mathcal{P}_\gamma)$ under appropriate decay assumptions at $\infty$.

The methods of moving planes was invented by Alexanderoff in the early 1950s. Later, it was further developed by Serrin [27], Gidas, Ni and Nirenberg [10, 11], Caffarelli, Gidas and Spruck [1], Chen and Li [2], Lin [14], Chen, Li and Ou [5] and many others. For more literatures on the classification of solutions and Liouville type theorems for various PDE and IE problems via the methods of moving planes or spheres, please refer to [3, 4, 6, 7, 12, 14, 15, 21, 28] and the references therein.

The main result in our paper is the following theorem.

**Theorem 1.1.** Assume $d \geq 9$ and $0 < \gamma \leq 1$. Suppose $u$ is a nonnegative classical solution of $(\mathcal{P}_\gamma)$ with $u(x) = O(|x|^{-\frac{d+4}{2}})$ as $|x| \to \infty$. Then, $u$ is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^d$ in the critical case $\gamma = 1$, in particular, the positive classical solution $u$ must assume the following form

$$u(x) = \mu \frac{d+4}{2} W(\mu(x - x_0)) \quad \text{for some } \mu > 0 \text{ and } x_0 \in \mathbb{R}^d,$$

where $W(x) = \sqrt{\frac{(d+2)(d-2)(d-4)!}{\pi^2 \Gamma(\frac{d+2}{2})}} \left( \frac{1}{1+|x|^2} \right)^{\frac{d+4}{2}}$. For $0 < \gamma < 1$, the unique nonnegative classical solution of $(\mathcal{P}_\gamma)$ is $u \equiv 0$ in $\mathbb{R}^d$. 

Remark 1.2. The decay assumption in Theorem 1.1 is necessary because we must ensure the integral in convolution \( \frac{1}{|x|} \ast |u|^2 \) is finite. One should observe that, by employing the Kelvin transforms, we only need to assume the weaker decay property \( u(x) = O\left( \frac{1}{|x|^d} \right) \) in Theorem 1.1 instead of \( u(x) = O\left( \frac{1}{|x|^{d-4}} \right) \) at \( \infty \). Our decay assumption is also weaker than the global integrability condition \( u \in L^{\frac{2d}{d-4}}(\mathbb{R}^d) \). We can see clearly from the proof that the conclusions in Theorem 1.1 still hold true under the integrability assumption \( u \in L^{2d/(d-4)}(\mathbb{R}^d) \).

The classification of the solutions to \((P_1)\) would provide the best constants and extremal functions for the corresponding Hardy-Littlewood-Sobolev inequality (see [16]).

We define the norm \( \|u\|_{L^V} := \| (V \ast |u|^2) |u|^2 \|_{L^1(\mathbb{R}^d)} \) with potential \( V = \frac{1}{|x|^4} \). For \( u \in H^2(\mathbb{R}^d) \), we have the following Hardy-Littlewood-Sobolev inequality (see [12, 16])

\[
\|u\|_{L^V} \leq S_d^{-1} \|\Delta u\|_{L^2},
\]

where the best constant \( S_d \) is given by

\[
S_d = \inf_{u \in H^2(\mathbb{R}^d), u \neq 0} \frac{\|\Delta u\|_{L^2}}{\|u\|_{L^V}}.
\]

Equation \((P_1)\) is the corresponding Euler-Lagrange equation for the minimization problem described in (1.1). By the uniqueness of the spherically symmetric positive solutions of the Euler-Lagrange equation \((P_1)\) derived in Theorem 1.1 and using the concentration-compactness arguments in [17, 18], we can obtain the following variational characterization of \( W \) (see [8, 17, 18, 22, 25]).

**Corollary 1.3.** For \( d \geq 9 \), all the minimizers for \( S_d \) make up a set \( \mathcal{M} \) of the form

\[
\mathcal{M} = \{ e^{i\theta} \mu^\frac{d-4}{2} W(\mu(x-y)) : \forall \theta \in (-\pi, \pi], \mu > 0, y \in \mathbb{R}^d \}.
\]

**Remark 1.4.** One can deduce from the definition of \( S_d \) and equation \((P_1)\) that

\[
\|\Delta W\|_{L^2} = S_d \|W\|_{L^V}, \quad \|\Delta W\|^2_{L^2} = \|W\|^4_{L^V},
\]

therefore, the best constant \( S_d \) for Hardy-Littlewood-Sobolev inequality (1.3) can be calculated explicitly as

\[
S_d = \|\Delta W\|^\frac{1}{2}_{L^2} = \|W\|_{L^V}.
\]

By further calculations, we can deduce from (1.6) the following corollary.

**Corollary 1.5.** The best constants in the Hardy-Littlewood-Sobolev inequality (1.3) is

\[
S_d = \frac{1}{2} \sqrt{(d+2)d(d-2)(d-4)} \left( \frac{(d-6)(d-8)}{(d-1)(d-3)} \right)^{\frac{1}{4}}.
\]
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Institute of Applied Mathematics, AMSS, Chinese Academy of Sciences, Beijing 100190, P. R. China

E-mail address: dmcao@amt.ac.cn

School of Mathematics and Systems Science, Beihang University (BUAA), Beijing 100191, P. R. China

E-mail address: weidai@buaa.edu.cn