Fashion and Homophily

Boyu Zhang  
School of Mathematical Sciences, Beijing Normal University, Beijing, 100875, China, zhangboyu5507@gmail.com

Zhigang Cao  
MADIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China, zhigangcao@amss.ac.cn

Chengzhong Qin  
Department of Economics, University of California, Santa Barbara, CA 93106, USA, qin@econ.ucsb.edu

Xiaoguang Yang  
MADIS, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China, xgyang@iss.ac.cn

Fashion, as a “second nature” of human beings, plays a quite non-trivial role not only in economy but also in many other areas, including even academia. We classify agents pursuing fashion into two types, conformists and rebels, who prefer to match and mismatch the majority action of their neighbors, respectively, and investigate their interactions through a social network. We are interested in how the structure of the social network affects the evolution of fashion, especially the emergence of fashion cycles. The conclusion is quite clean: homophily, in general, inhibits the emergence of fashion cycles. That is, to have the emergence of fashion cycles, conformists and rebels should be adequately mixed. This improves the insightful observation of Simmel (1904) that heterogeneity of consumers is essential to fashion. To establish the main result, a novel partial potential analysis is invented and recent approximation skills, which are developed by theoretical biologists and are in essence an elaborate mean-field method, are applied. Technically, this network game (referred to as the fashion game) can also be taken as a network extension of three elementary games: the coordination game, the anti-coordination game, and Matching Pennies. Through partial potential analysis, we show that the fashion game always possesses a pure Nash equilibrium if each conformist has no less conformist neighbors than rebel ones, improving a result of Jackson (2008). Through approximation skills, we transform the stochastic best response dynamic of the fashion game into a system of ordinary differential equations and give a fairly complete analysis of it. Simulations show that this approximation is quite reliable.

Key words: network games; fashion cycle; homophily; Matching Pennies; partial potential analysis

1. Introduction

Although the world’s wealth is distributed quite unevenly, the unprecedented overall affluence is an undeniable fact. For most people living in developed countries, as well as those in developed areas of developing countries, their basic physical needs in life are easily met. They live mainly not for personal comfort, but for social comfort (Scitovsky, 1986). Consequently, compared with physical
value, social value becomes a more and more crucial factor when people consider whether to buy a product (van Nes, 2010). Suppose there are two kinds of products that have the same functions, similar qualities and performances, but one is very “in” and the other is “out”. Then it is not surprising that the “in” product can be much more expensive than the “out” one, because its social value is much higher. Social value covers in general “the image of the product, the appreciation by others, the sense of luxury and the provision of status” (van Nes, 2010). However, how bright the image of a product is, whether you can get appreciation through consuming this product, and whether it will give you the sense of luxury or the proper social identity, are all mostly determined by how other people view this product. The overall consumption related viewpoints or collective taste of the society can be summarized very well in the current fashion trends.

Fashion plays such a key role in producing the total value of a product that it is ranked second only to technology, because as technology contributes most to the physical value of a product, fashion contributes most to social value (note that this is valid only for consumer goods, not for capital goods). The complementary relation between technology and fashion can also be observed in a perhaps even more important fact that while technology accelerates, and sometimes creates, supply, it is fashion that influences more directly on demand, through shaping consumers’ tastes and consequently their purchasing behavior.

Fashion has a great impact on environment. People’s discarding behavior is consistent with their purchasing one. When they consider whether to discard a product, the reason that it fails to work is becoming less and less often, and an increasingly frequent reason is that it is not fashionable any longer. It has been confirmed by research that for many kinds of products, e.g., furniture and appliances (Anderson, 1999; Cooper and Mayers, 2000), electronic products (Slade, 2006), a significant proportion of them still function very well when discarded. The chief reason is that they are “out of fashion”. Fashion, “the key hazard for durability of consumer goods”, has been hand in hand with the throwaway culture ever since industrialization (Stahel, 2010). Overconsumption is the natural result, huge amount of unnecessary wastes are produced, and limited natural resources are squandered.

Fashion is also important to economy. There are at least two reasons. First of all, fashion is a huge industry. Despite all the complications to give a satisfactory definition of the fashion industry, its global market size is estimated at several hundred billion US dollars (Okonkwo, 2007; Euromonitor International, 2011; Yoganarasimhan, 2012b). Second of all, fashion stimulates economic growth. Planned obsolescence has long been believed by governments as an efficient way to stimulate economic growth, at least in economic crises when production fails to be able to continue. From the Great Depression to the global recession in 2008, consumption stimulating policies, which encouraged consumers to replace still functional products, were introduced to rescue the market
Packard, 1963; Slade, 2006). Fashion, as a constant consumption stimulator, plays a similar role. Although this theory is greatly debatable, it is certain that the impact of fashion on economic growth is considerable.

Not only in environment and economy, but also in many other areas of the society, e.g., education, politics, art, and even science and academic research, fashion is a factor that cannot be neglected (Blumer, 1969). Originating in the before industrial society almost at the very beginning of civilization, and flourishing in the industrial society, fashion is now playing a more and more crucial role in almost every corner of the post industrial society. A recent research shows that charitable donation among celebrities is also a highly subject of fashion (Schweitzer and Mach, 2008). In fact, as claimed by Svendsen (2006), fashion has been one of the most influential phenomena and has become almost ‘second nature’ to us.

In this paper, we shall study the evolution of fashion through a heterogeneous network game model. Before moving to the motivation of this paper, let’s see what fashion means precisely.

On what is fashionable, interestingly, there are two different, in fact almost opposite, viewpoints that are both extremely popular. We often hear that “This year’s fashion color is black”, which indicates that conforming to the majority is fashionable. However, we also hear that, perhaps equally often, “Wow, Lady Gaga is fashionable!”, which implies that being distinctive is the essence. In the dictionary of Merriam-Webster, there are two definitions for fashion, reflecting the two opposite opinions. One is that “a prevailing custom, usage, or style”, and the other is “a distinctive or peculiar and often habitual manner or way”. Following Jackson (2008), we shall call the above two kinds of people, holding opposite opinions on fashion, conformists and rebels, respectively. The model explored in this paper depends crucially on classification of people into the above two types.

1.1. Motivation

Compared to technology, the other main driving force in market and society, fashion is much more intangible and unpredictable. This is quite understandable, because as technology reflects the ability of human beings, fashion reflects our dreams. The development of technology has an “arrow”: like time, it keeps advancing, and very few people doubt the positive role of technology. In contrast, there is no clear tendency for the evolution of fashion, and the multi-fold roles that fashion plays are rather complicated. Like economy has ups and downs, fashion comes and goes, and after usually an unexpected period of time, it may come back again. This phenomenon, supported by numerous daily observations, e.g. fashion color and length of women’ skirts, is called fashion cycle. Fashion cycle is usually taken as the most extraordinary and the most “mysterious” part of fashion. It is so crucial to understand fashion that it could be deemed as the “pulse” of it. If fashion
were not in constant flux, then its impact on environment and economy would have disappeared, because people would not replace their goods at all, until they are completely worn out.

Not surprisingly, trying to understand the logic of fashion cycle is one of the key focuses of previous studies (cf. Pesendorfer, 1995; Young, 2001; Yoganarasimhan, 2012a). However, as pointed out by Yoganarasimhan (2012b), systematic research is still quite lacking, and even “[...] how to detect fashion cycles in data has not been studied.”

In this paper, we are particularly interested in the theoretical problem of “what causes fashion cycle”. It is safe to say that fashion cycle is caused ultimately by a combined force of consumers and producers. However, the concrete process may be quite complicated. In the classical model of Pesendorfer (1995), fashion cycle is completely controlled by a monopolist producer. Although that everything is decided by a powerful producer is obviously unrealistic, because very rarely can a producer be so powerful, this theory provides us new understandings of the fashion cycle on the producer side. In this paper, we shall attack this problem sheerly from the consumer side. We investigate how the structure of consumers’ social interactions affects the evolution of fashion, and particularly the emergence of fashion cycles.

Since fashion comes from comparisons between people, and the range that people compare is almost always confined to their friends, relatives, colleagues, and neighbors, that fashion works through a social network is the most natural thing. The model we apply, which is referred to as the fashion game, can be taken as the network extension of Matching Pennies (to be introduced in Section 2). Note first that Matching Pennies is indeed a special case of the fashion game, because the row player can be interpreted as a conformist, and the column player a rebel (Figure 1 in Section 2). As far as we know, the formulation of the fashion game should be attributed to Jackson (2008, P.271). Young (2001, P.38) also studies a very similar model, but the social network is not expressed explicitly.

1.2. Contribution

In a companion paper, it is mentioned that heterogeneity of consumers, a factor pointed out originally by Simmel (1904), might play an important role in the emergence of fashion cycles (Cao and Qin et al., 2013). To be precise, Simmel wrote that “Two social tendencies are essential to the establishment of fashion, namely, the need of union on the one hand and the need of isolation on the other. Should one of these be absent, fashion will not be formed—its sway will abruptly end.” This argument will be formally proved: it can be easily shown that the two extreme cases of the fashion game with either only conformists or with only rebels are both exact potential games (Theorem 2). Thus, under the very natural best response dynamic (BRD for short), a pure Nash
Equilibrium (PNE for short) will eventually be reached in both cases, and fashion cycle has no chance to occur.

We deepen the above argument by providing a more subtle structural factor, i.e. heterophily of consumers’ social interaction structures (to be extensively investigated in Section 3). Intuitively speaking, we argue that to have the emergence of fashion cycles, conformists and rebels must be adequately mixed. To establish the main result, BRD is taken as the underlying dynamic process, and two approaches are applied: one is rigorous and the other approximate.

In the first approach, we take the empirical proposition that fashion cycle exists as roughly equivalent to the theoretical one that PNE does not exist. On the one hand, when there is no PNE at all, we cannot expect to have a steady state where no agent has incentive to deviate. So fashion keeps in flux and fashion cycles are likely to emerge (note that there are only a finite number of states). On the other hand, when PNE does exist, as long as perturbations are introduced (i.e. agents err with a tiny probability \( \epsilon \) in each response), the system will almost surely keep in a steady state in the long run (as \( \epsilon \) tends to zero), i.e. fashion cycles are not likely to occur. By applying the standard potential analysis (Monderer and Shapley, 1996) and a novel partial potential analysis (Appendix EC.4), the PNE existence problem is investigated from both the homophily side (Theorem 2, Theorem 3) and the heterophily side (Theorem 1). The partial potential analysis may also have some technical interest of its own. It bridges potential games and PNE possessing games, providing a unification framework (Subsection EC.4.4).

In the second approach, we transform the stochastic BRD of the fashion game approximately into a system of ordinary differential equations (ODE for short), and very naturally periodic solutions are taken as fashion cycles. Technically, this transformation relies crucially on recent developments of the pair approximation and diffusion approximation skills in the field of theoretical biology (Ohtsuki et al., 2006; Ohtsuki and Nowak, 2006, 2008). This skill, in essence, is an elaborate implementation of the mean-field method. Numerical simulations verify that this approximation is quite reliable (Figure 4, Figure 5). A nice property of the fashion game is that the dynamic stability of the corresponding ODE can be completely characterized (Table 1). In particular, Theorem 5 points out that heterophily on average is a necessary condition for the emergence of fashion cycles.

The two approaches are complementary to each other: the first one is rigorous but only deals with very special cases, while the second one is approximate but attacks the general case. Results derived by the two approaches are perfectly consistent, all supporting our main conclusion that homophily, in general, inhibits the emergence of fashion cycles.

1.3. Organization

Organization of the rest paper is as follows. In the next subsection, we provide a very brief literature review. Subsection 1.5 introduces the concepts of homophily and heterophily. Section 2
presents the formal model and its relation with Matching Pennies (as well as coordination game and anti-coordination game). Section 3 is for the measuring of homophily and heterophily. Several homophily/heterophily indices are introduced. Section 4 gives the standard potential analysis in terms of homophily. Section 5 and Section 6 are the main parts of this paper, providing the partial potential analysis and the approximation analysis, respectively. Section 7 gives some marketing implications of our results, and Section 8 concludes this paper with more interpretations of this model. All non-straightforward proofs and technical details are postponed to the Electronic Companion.

1.4. Related Work

Jackson (2008) observes that if every agent has no less conformist neighbors than rebel ones, then PNE is guaranteed. Theorem 3 of this paper is actually an improvement of Jackson’s observation. Cao and Yang (2011) mainly concentrate on the computational side of the fashion game. They prove that checking whether a fashion game has a PNE is in general NP-hard, and discuss several easy special cases. The NP-hardness result indicates that predicating fashion trends might be impossible. This is supported by industrial practices from fast fashion (See Cao and Qin et al. (2013) for the example of Zara). In the companion paper, Cao and Qin et al. mainly investigate the evolution of fashion on several special network structures, i.e. lines, rings, and stars. The dynamics they apply are also BRDs, both the simultaneous one and the sequential one. For the simultaneous BRD, the concept of limit cycle, from the filed of discrete dynamic systems, is borrowed to study fashion cycle. Through simulation, Cao and Gao et al. (2013) find that in general people can reach an extraordinarily high level of cooperation via stochastic BRD.

For fashion games with conformists only, Goles (1987) shows that simultaneous BRD always converges to a cycle of length at most 2 (see also Berninghaus and Schwalbe, 1996). This result is recently extended to more general models with arbitrary thresholds (Adam et al., 2013). Cannings (2009) notices that the result of Goles holds for fashion games with rebels only too. He also observes that when the network structure is complete and the underlying dynamic is simultaneous BRD, the only possible limit cycle lengths are 1, 2 and 4. For an extension of one side of the fashion game with only rebels, see Bramoullé et al. (2004), Bramoullé (2007), and Cao and Yang (2012).

Pesendorfer (1995) and Bagwell and Bernheim (1996) are two of the most influential papers studying fashion in the field of economics. The main differences between their models and the fashion game in this paper are that, in their papers, (i) consumers are classified into upper class and lower class, who possess different amount of resources, (ii) producers are involved, and (iii) no explicit social network is considered.
Our research falls into the booming field of network games (see Goyal, 2007; Vega-Redondo, 2007; Jackson, 2008; Jackson and Zenou, 2014). It can be taken as a new application of this field. What’s more, the fashion game also has its theoretical interest in this field, namely, it is a mixed model of a network game with strategic complements and a network game with strategic substitutes, the two most extensively studied classes of network game models (See e.g. Bramoullé (2007), Bramoullé and Kranton (2007), Hernández et al. (2013). See also the review in Jackson (2008), and Jackson and Zenou (2014)). In fact, for conformists, it exhibits a simple form of strategic complement, and for rebels a simple form of strategic substitute works. In a recent model of Hernández et al. (2013), strategic complements and strategic substitutes can be both embodied too. However, for a particular setting of parameters, only one effect works. In our model, except for the two extreme cases, the two effects work simultaneously.

1.5. Homophily and Heterophily

Homophily, a concept extracted from the daily observation that “birds of a feather flock together”, is attracting more and more attention recently. Intuitively, this concept means that agents are prone to interact more with those from the same type or similar to them. It has been shown by numerous evidences that this is indeed the case in the real world when type refers to age, gender, race, religion, and education etc. (McPherson et al., 2001). The opposite phenomenon is called heterophily.

The importance of homophily comes not only from its wide existence, but also from the fact that its existence can have tremendous impacts on people’s behaviors. For instance, homophily may slow the convergence speed in social learning and best-response processes (Golub and Jackson, 2012). It is recently shown that homophily can also serve as a similar role as the classical “rich get richer” mechanism in the growth of networks that produces the fundamental power-law distribution (Papadopoulos et al., 2012).

2. Model

Formally, each fashion game is represented by a system \( G = (N, E, T, A, (u_i(\cdot))_{i \in N}) \), where

- \( N = \{1, 2, \cdots, n\} \) is the set of agents, who are also interchangeably referred to as players and represented by nodes of the network;
- \( E \subseteq N \times N \) is the set of (undirected) edges. For agents \( i, j \in N \), they are neighboring to each other if and only if \( ij \in E \). \( \forall i \in N, N_i(G) \) is the set of \( i \)'s neighbors in \( G \), and \( d_i(G) \) her degree, i.e. the number of \( i \)'s neighbors;
- \( T \equiv [T_i], i \in \{C, R\}^N \) is the configuration of types. For each agent \( i \in N \), \( T_i \) is her type: \( T_i = C \) means that \( i \) is a conformist, and \( T_i = R \) a rebel;
A = \{0, 1\} is the identical action set for all players. We also use \( A^N \) to denote the set of pure action profiles, and \( a \equiv [a_i]_i \in A^N \) a pure action profile. As usual, \( a_{-i} \) is the partial action profile of players other than \( i \).

\( \forall i \in N, u_i(\cdot) : A^N \to \mathbb{R} \) is her utility function. Recall that our basic assumption is that conformists like her neighbors who choose the same action as she does, but dislikes those who choose the different action, while rebels’ preferences are the opposite. So we define \( L_i(a_i, a_{-i}) \) as the set of neighbors of player \( i \) that she likes in \( a \equiv (a_i, a_{-i}) \), i.e.

\[
L_i(a_i, a_{-i}) = \begin{cases} 
\{ j \in N_i(G) : a_j = a_i \} & \text{if } T_i = C \\
\{ j \in N_i(G) : a_j \neq a_i \} & \text{if } T_i = R 
\end{cases}
\]

And similarly, \( D_i(a_i, a_{-i}) = N_i(G) \setminus L_i(a_i, a_{-i}) \) is the set of neighbors that player \( i \) dislikes in \( a \).

The payoff of each player \( i \) is simply defined as the number of neighbors she likes minus that she dislikes, i.e.

\[
u_i(a_i, a_{-i}) = |L_i(a_i, a_{-i})| - |D_i(a_i, a_{-i})|.
\]

Note that in the above definition we actually assume that actions in our model, which can be interpreted as goods, have only social values but no physical values. This is indeed the essence of fashion goods. As noticed by Jackson (2008), the fashion game is an extension of Matching Pennies. In fact, the special case of the fashion game where there are only two players, one conformist and the other rebel, who are connected to each other, is exactly Matching Pennies. Also, the fashion game takes the coordination game and the anti-coordination game as special cases: (i) if the game consists of only two players who are conformists and connected, it is coordination game; (ii) in contrast, if the game consists of only two players who are rebels and connected, it is anti-coordination game.

The relations between the fashion game and the three elementary games are summarized in Figure 1.

More generally, we can say that the fashion game is the network extension of the three two-player games, which are also referred to as base games. To be precise, as in the network extension of any two-player game, we assume that (i) each agent plays once with each of her neighbors (one of the three base games according to their types), (ii) for any agent, in all the base games that she plays, she should take the same action, (iii) the total utility of each player is simply the sum of the payoff that she gets from all the base games she plays. Using payoff settings in Figure 1, it can be checked that the fashion game with utility function (1) is indeed equivalent to the above “decomposition”.

3. Measuring Homophily and Heterophily

We note first that all definitions and results in this section are valid for networks with two types of nodes (agents). For the sake of their applications in later sections, we present them in terms of the fashion game. Above all, we need several more notations.
Figure 1  Three base games. (1) Coordination Game (left): conformist V.S. conformist; (2) Matching Pennies (middle): conformist V.S. rebel. (3) Anti-coordination Game (right): rebel V.S. rebel. Conformists are represented by circles, and rebels by triangles.

Given a fashion game $G$, we use $C(G)$ to denote the set of conformists, and $R(G)$ the set of rebels. $n(G)$ is the total number of players, $c(G)$ that of conformists, and $r(G)$ of rebels. Straightforwardly, $N = C(G) \cup R(G)$ and $n(G) = c(G) + r(G)$. We also use $f_c(G)$ and $f_r(G)$ to denote the fractions of conformists and rebels, respectively, among the whole population $N$. We write $N^c_i(G)$ to denote the set of conformist neighbors of player $i$ in $G$, and $N^r_i(G)$ that of rebel neighbors. For each player $i \in N$, $d^c_i(G)$ and $d^r_i(G)$ are the numbers of conformist neighbors and rebel neighbors of $i$, respectively. Finally, we use $k_{CC}(G)$, $k_{RR}(G)$ and $k_{CR}(G)$ to denote the total numbers of CC, RR and CR edges in $G$, respectively.

3.1. Basic Concepts

The following indices measuring homophily levels are very natural, and have been widely used in literature (e.g., Currarini et al., 2009).

**Definition 1 (Homophily Indices).** Given a fashion game $G$, its conformist homophily index and rebel homophily index, denoted by $h_c(G)$ and $h_r(G)$, respectively, are defined as follows

$$h_c(G) = \frac{2k_{CC}(G)}{2k_{CC}(G) + k_{CR}(G)},$$

$$h_r(G) = \frac{2k_{RR}(G)}{2k_{RR}(G) + k_{CR}(G)}.$$

A prominent advantage of the above definition is that the “macro” measures can be derived naturally from “micro” ones. Given a fashion game $G$, for any player $i$, her individual homophily index is defined as

$$h_i(G) = \begin{cases} 
\frac{d^c_i(G)}{d_i(G)} & \text{if } i \in C(G) \\
\frac{d^r_i(G)}{d_i(G)} & \text{if } i \in R(G) 
\end{cases}.$$  

(2)

The relations between the micro and macro measures can be stated in the following straightforward equations

$$h_c(G) = \frac{\sum_{i \in C(G)} d_i(G) h_i(G)}{\sum_{i \in C(G)} d_i(G)},$$
\[ h_R(G) = \frac{\sum_{i \in R(G)} d_i(G) h_i(G)}{\sum_{i \in R(G)} d_i(G)} . \]

The above relations tell us that the conformist/rebel homophily index of \( G \) is exactly the weighted average of the individual homophily indices of all the conformists/rebels, where the weights are their degrees. The proposition below is also straightforward, because for any positive number \( a \), the function \( x/(a + x) \) is increasing for non-negative \( x \).

**Proposition 1.** \( h_C(G) \geq h_R(G) \) if and only if \( k_{CC}(G) \geq k_{RR}(G) \).

When we talk about whether a fashion game satisfies homophily or not w.r.t. a particular type of players, we can simply compare the corresponding homophily index with its fraction in the whole population. The definition below is standard (e.g., Currarini et al., 2009).

**Definition 2 (Relative Homophilies/Heterophilies).** Let \( G \) be a fashion game. We say that \( G \) satisfies conformist (rebel) relative homophily if \( h_C(G) > f_C(G) \) (\( h_R(G) > f_R(G) \)). On the other hand, we say that \( G \) satisfies conformist (rebel) relative heterophily if \( h_C(G) < f_C(G) \) (\( h_R(G) < f_R(G) \)). The remaining case, \( h_C(G) = f_C(G) \) (\( h_R(G) = f_R(G) \)), is called conformist (rebel) baseline homophily.

In the rest of this paper, when no confusion is incurred, the variable \( G \) in all the notations may be omitted.

### 3.2. Possible Inconsistencies

An obvious complication is that there are two indices, the conformist one and the rebel one. Hence it is immediate to ask whether they are consistent. By consistency, we expect naturally the three properties below: (i) \( h_C = 1 \) if and only if \( h_R = 1 \), (ii) \( h_C = 0 \) if and only if \( h_R = 0 \), (iii) \( h_C > f_C \) if and only if \( h_R > f_R \).

Property (i) holds trivially, because \( h_C = 1 \) and \( h_R = 1 \) both mean that every agent has connections only with those of the same type (see more discussions of the next subsection). Properties (ii) and (iii), however, do not hold in general. Figure 2 illustrates a counter-example to both of them. In fact, we have the following general result.

**Proposition 2.** For any three rational numbers \( f_C \in (0,1), h_C, h_R \in [0,1) \), there exists a fashion game with conformist fraction \( f_C \), conformist homophily index \( h_C \), and rebel homophily index \( h_R \).
3.3. More Homophily Concepts

According to Proposition 2, we can say that the three parameters, $f_C$, $h_C$, and $h_R$, are (almost) independent of each other. So when we say a fashion game satisfies relative homophily (heterophily), it is safe to require that this is true for both types. However, we find the following weaker concept more useful.

**Definition 3 (Average Homophily Index).** Given a fashion game $G$, its average homophily index, denoted by $h(G)$, is defined as the average of its conformist homophily index and the rebel one, i.e.

$$h(G) = \frac{h_C + h_R}{2}.$$

**Definition 4 (Homophily/Heterophily on Average).** We say that a fashion game $G$ satisfies homophily on average if $h(G) > 1/2$ and heterophily on average if $h(G) < 1/2$.

It turns out that homophily/heterophily on average can be characterized very well by a simple relation between numbers of inner-group and cross-group edges.

**Proposition 3.** A fashion game $G$ is homophilic on average if and only if $k_{CR}(G) < k_{CC}(G) + k_{RR}(G)$, and heterophilic on average if and only if $k_{CR}(G) > k_{CC}(G) + k_{RR}(G)$.

The following concepts are also useful.

**Definition 5 (Complete Homophily/Heterophily).** We say that a fashion game $G$ satisfies complete homophily if $h(G) = 1$ and complete heterophily if $h(G) = 0$.

In general, it is difficult to estimate the numbers of the three types of edges if we only know the average homophily index. However, when $G$ is completely homophilic, it must be true that $k_{CR}(G) = 0$. That is, no conformist has any connection (direct or indirect) with a rebel, and vice versa. And when $G$ is completely heterophilic, we must have $k_{CC}(G) = k_{RR}(G) = 0$. That is, no conformist has any conformist neighbor, and no rebel has any rebel one. Using the terminology of
graph theory, complete heterophily means that the underlying graph is bipartite, with one side of all conformists and the other side of all rebels. The proposition below is evident.

**Proposition 4.** Let $G$ be a fashion game. If conformists and rebels both exist, i.e. $0 < f_C < 1$, then complete homophily/heterophily implies relative homophily/heterophily, which further implies homophily/heterophily on average.

For regular graphs, which are defined as those that all nodes have identical degrees, we have the following nice property.

**Proposition 5.** Let $G$ be a fashion game. If the underlying graph of $G$ is regular, then the three concepts, conformist relative homophily, rebel relative homophily and homophily on average, are equivalent.

Thanks to the above proposition, it is definite that a fashion game with a regular underlying graph satisfies either relative homophily, or relative heterophily, or baseline homophily.

4. Preliminary Analysis

Like Matching Pennies, the fashion game has an obvious mixed Nash equilibrium: each agent plays 0 and 1 equally likely. In this paper, we are more interested in pure strategy Nash equilibria (PNE for short). An (pure) action profile $a^* \in \{0,1\}^N$ is a PNE if players have no incentive to change their actions. Since $u_i(0,a_{-i}) + u_i(1,a_{-i}) = 0$ for all $a_{-i} \in \{0,1\}^{N\setminus\{i\}}$, we can see that $a^*$ is a PNE if and only if

$$u_i(a_{-i}^*,a_{--i}^*) \geq 0, \forall i \in N.$$ 

Given an action profile $a$, we say that agent $i$ is *satisfied* if $u_i(a) \geq 0$. Otherwise, she is called *unsatisfied*.

**Proposition 6.** The fashion game must have an even number of PNEs.

Proof to the above proposition is straightforward, because it is an immediate result of our payoff settings that the two actions 0 and 1 are symmetric. In fact, if $a^*$ is a PNE, then $\bar{a}^*$ must also be a PNE, where $\bar{a}_i^* = 1 - a_i^*$ for all $i \in N$. The above proposition embodies the essence of fashion as *symbolic consumption*. That is, what a single player consumes does not matter, what matters is the comparison with her neighbors. Therefore, if all people switch their actions, nothing essential is changed.
4.1. The Complete Heterophily Case

Existence of PNE, in general, cannot be guaranteed. For instance, as long as the fashion game has a conformist and a rebel who are linked to each other but have no other. If we require the network to be connected, things cannot be improved at all. Below is a more general negative result.

**Theorem 1.** Let $G$ be a fashion game. If $G$ satisfies complete heterophily, i.e. $h(G) = 0$, and at least one agent has an odd degree, then $G$ does not possess any PNE.

Note that in the above proposition, the second condition (oddness of at least one player’s degree) is quite weak. Therefore, for almost all networks, the condition for the above proposition is equivalent to a requirement of complete heterophily. To put it another way, complete heterophily implies almost always the inexistence of a PNE. It can be shown that if the underlying graph is a line, then complete heterophily is the only case that PNE does not exist. For rings (where no player has an odd degree), complete heterophily is a necessary but not sufficient condition for the in-existence of PNE (Cao and Yang, 2011).

4.2. The Complete Homophily Case

It is natural to ask the other polar case, i.e complete homophily. We are happy to see that this case always possesses a PNE.

**Theorem 2.** Let $G$ be a fashion game. If $G$ satisfies complete homophily, i.e. $h(G) = 1$, then $G$ is an exact potential game, and thus possesses at least one PNE.

Theorem 2 is implied by a more general result of Blume (1993), which states that any symmetric social interaction game possesses an exact potential function (see also Bramoullé, 2007). In fact, $h(G) = 1$ says that no inter-group connection exists between conformists and rebels, and hence all interactions are symmetric. As a corollary, network coordination games and network anti-coordination games, which are two special fashion games satisfying complete homophily, are both exact potential games. Since the rigorous proof to the above theorem is not hard (the potential function can be defined as a half of the total utilities of all players), we present it in EC.2.2 for the convenience of the reader.

It is well known that the set of PNEs of a network coordination game forms a lattice (e.g., Jackson and Zenou, 2012). In contrast, it is never noticed before that a PNE in the network anti-coordination game corresponds to an *unfriendly partition*, a concept that has been actively studied by graph theorists (see e.g., Bollobás and Scott, 2002).
5. Partial Potential Analysis

Standard theory of potential games tells us not only that PNE is guaranteed for the completely homophilic fashion game, but also that, from any initial action profile, PNE can be reached through the best (in fact better) response dynamics with arbitrary deviating orders (Monderer and Shapley, 1996). However, the potential argument fails to work for the case that PNE exists but might not be reached through best response dynamics. In this section, we develop a novel partial potential analysis to investigate the existence of a PNE for non-potential games.

5.1. Basic Idea

The basic idea of partial potential analysis is as follows. We let a part of players fix their actions and let the remaining ones deviate whenever unsatisfied (one deviation per step). If we can find an “arrow” for this process (i.e. a partial potential function, which has a finite number of values), and thus it cannot continue forever, then we will definitely arrive at a pure action profile where all the “free” players are satisfied. The larger the free set is, the better the final action profile will be. What’s more, if we can choose a partial action profile for the fixed ones such that they are always satisfied whatever the free ones choose, then we will arrive at a PNE.

5.2. Strong Homophilies

To present our main result, the following two definitions are needed. Recall first the definition of individual homophily index in (2).

**Definition 6 (Strong Conformist Homophily).** Let $G$ be a fashion game. If each conformist has no less conformist neighbors than rebels ones, i.e.

\[ h_i(G) \geq 0.5, \forall i \in C, \]

then we say that $G$ satisfies strong conformist homophily.

**Definition 7 (Strong Rebel Homophily).** Let $G$ be a fashion game. If there exists a partition of the rebel set $R$, $\{ R_1, R_2 \}$, such that

\[ |N_i(G) \cap R_2| \geq 0.5|N_i(G)|, \forall i \in R_1, \]

\[ |N_j(G) \cap R_1| \geq 0.5|N_j(G)|, \forall j \in R_2, \]

then we say that $G$ satisfies strong rebel homophily.

The proposition below shows that the two homophilies are weaker than complete homophily but almost always stronger than homophily on average.
Proposition 7. Let $G$ be a fashion game.

(a) If $G$ satisfies complete homophily, i.e. $h(G) = 1$, then it satisfies both strong conformist homophily and strong rebel homophily.

(b) If $G$ satisfies both strong conformist homophily and strong rebel homophily, then $h(G) \geq 0.5$.

5.3. Main Result

Jackson (2008) observes that when (i) each conformist has an individual homophily index of at least 0.5 and (ii) each rebel has an individual homophily index of at most 0.5, then PNE exists. In fact, it can be checked that all conformists choose 0 and all rebels choose 1 is a PNE. The conditions used by Jackson are stronger than strong conformist homophily. To be precise, the first part of his condition is exactly strong conformist homophily. Half of the following theorem is an improvement of Jackson’s result, showing that the condition that rebels should have an upper bound of 0.5 on their individual homophily indices can be actually removed.

Theorem 3. Let $G$ be a fashion game. If $G$ satisfies either strong conformist homophily or strong rebel homophily, then PNE exists.

As to PNE existence, the above theorem is also stronger than Theorem 1, because $h(G) = 1$ implies both strong conformist homophily and strong rebel homophily, as shown in Proposition 7(a). However, Theorem 1 still has its right to exist, because it tells us more: the complete homophily case of the fashion game is an exact potential game, which is not implied in Theorem 3.

Correctness of Theorem 3 follows naturally from the general framework of partial potential analysis introduced in EC.4 (note that this framework is for general normal form games). The rough idea behind that proof has already been sketched in Subsection 5.1. To be more specific, when $G$ satisfies strong conformist homophily, then we can let all the conformists choose 0. So the conformists are always satisfied whatever the rebels choose, because each of them has already at least a half of their neighbors take the same action as they do. Then we let the rebels take arbitrary initial actions and deviate in an arbitrary order whenever unsatisfied. Just as we have shown that the fashion game with all rebels is a potential game, the above reduced game (with all conformists choosing 0) can also be shown to be a potential game. Using this property, we know that the deviating process of rebels will eventually stop at a state where all the rebels are satisfied. Since all the conformists are satisfied too, we actually arrive at a PNE. The proof to the second part of Theorem 3 is analogous, and the only point that should be noticed is that when $G$ satisfies strong rebel homophily, with the rebel partition $\{R_1, R_2\}$ as in the definition, we can let rebels in $R_1$ choose 0 and those in $R_2$ choose 1, then all the rebels are always satisfied whatever the conformists choose.
In general, the fashion game is not an exact potential game in neither of the two cases in Theorem 3. In fact, using Corollary 2.9 of Monderer and Shapley (1996), it can be checked that the fashion game is definitely not an exact potential game when there is at least one conformist-rebel edge in the underlying graph. What’s more, as shown in Figure EC.2, fashion games satisfying conditions in Theorem 3 may not even be ordinary potential games. This means that the partial potential analysis is a substantial extension of the potential analysis, and thus it can be used to prove PNE existence for non-potential games.

6. Approximation Analysis

In this section, we turn our attention from potential and partial potential analyses to approximation analysis. The purpose is still to investigate the relation between homophily and fashion cycles. We need first to describe the underlying updating process and then reformulate it into a system of ordinary differential equations (ODE) by applying pair approximation and diffusion approximation.

6.1. The Stochastic BRD

The dynamic is still based on BRD, but deviations are random. In each time step, exactly one player is chosen to update, and the chances are equal for all players. Say player $i$ is picked out now. Let $a \in \{0, 1\}^N$ be the current action profile, then player $i$ switches her action with a certain probability if she is currently unsatisfied, i.e. $u_i(a) < 0$, but definitely does not switch if satisfied, i.e. $u_i(a) \geq 0$.

To be precise, denote $v_i(a) = u_i(a)/k_i$ as the normalized utility function of player $i$, which is also referred to as the satisfaction degree, and define the switching probability by $\phi(v_i)$, which is zero when $v_i \geq 0$, and positive when $v_i < 0$. We also require that $\phi(\cdot)$ is differentiable and decreasing at non-positive values, i.e. $\phi'(v) < 0$, for $v \leq 0$. Thus, the more a player is unsatisfied with the current state, the more likely s/he switches. Intuitively, $\phi(\cdot)$ is a smoothed generalization of the best response function. In the rest of this paper, the above process will be referred to as SBRD (Stochastic BRD).

SBRD may stop at a PNE where no player has incentive to deviate. Therefore, each PNE corresponds to an absorbing state of the Markov process. In a game without PNE, e.g., Matching Pennies, SBRD leads to a perpetual fluctuation. However, even if the game has at least one PNE, convergence of SBRD cannot be guaranteed (see an example in Figure EC.3).

Since we cannot expect to predict the long run behavior of SBRD, because calculating the absorbing states is by no means easier than testing the existence of PNEs, which is hard (Cao and Yang, 2011), we focus on the short run behavior by reformulating SBRD to ODE. To accomplish this, the key tricks we apply are pair approximation and diffusion approximation.
6.2. From SBRD to ODE

Given an action profile, we denote the set of conformists using action \( a \in \{0, 1\} \) by \( C^a \) and their number by \( n_i^a \), and rebels taking action \( a \) by \( R^a \) and their number by \( n_i^a \). Thus, the proportions of \( C^1 \) in \( C \) and \( R^1 \) in \( R \) are, respectively, \( x = \frac{n_i^C}{n_i} \) and \( y = \frac{n_i^R}{n_i} \). We shall concentrate on the evolution of the population state \((x, y)\).

Under the pair approximation, individual homophily indices, as defined in (2), are approximated by the overall homophily indices. For given \((x, y)\) and \((f_C, h_C, h_R)\), the average satisfaction degrees for \( C^1, C^2, R^1 \) and \( R^2 \) are respectively

\[
\begin{align*}
    v_{C1} &= h_C(2x - 1) + (1 - h_C)(2y - 1), \\
    v_{C2} &= h_C(1 - 2x) - (1 - h_C)(1 - 2y), \\
    v_{R1} &= (1 - h_R)(1 - 2x) + h_R(1 - 2y), \\
    v_{R2} &= (1 - h_R)(2x - 1) + h_R(2y - 1).
\end{align*}
\]

(3)

Notice first that when \( f_C = 1 \), \( y \) has no definition at all. However, \( f_C = 1 \) implies that \( h_C = 1 \), and hence the coefficient of \( y \) in this case is always zero. Similar logic also works for the case \( f_C = 0 \), where \( x \) has no definition. So we assume w.l.o.g. in the rest of this paper that (3) is meaningful for all \( f_C \in [0, 1] \).

It can also be observed from (3) that \( v_{C1} = -v_{C2} \) and \( v_{R1} = -v_{R2} \). Therefore, transition probabilities of SBRD are

\[
\begin{align*}
    P(n_i^C \rightarrow n_i^C + 1) &= f_C(1 - x)\phi(v_{C2}), \\
    P(n_i^C \rightarrow n_i^C - 1) &= f_C x\phi(v_{C1}), \\
    P(n_i^R \rightarrow n_i^R + 1) &= (1 - f_C)(1 - y)\phi(v_{R2}), \\
    P(n_i^R \rightarrow n_i^R - 1) &= (1 - f_C)y\phi(v_{R1}).
\end{align*}
\]

(4)

where \( n_i^C \rightarrow n_i^C + 1 \) means that the population leaves state \((x, y)\) and enters \((x + \frac{1}{nf_C}, y)\). The meanings of the other three notations are similar.

Applying the diffusion approximation (Traulsen et al., 2005, 2006), the above process could be described by the stochastic replicator-mutator equations

\[
\begin{align*}
    \frac{dx}{dt} &= (1 - x)\phi(v_{C2}) - x\phi(v_{C1}) + \sum_{i=1}^3 c_{i1}(x, y)\xi_i(t), \\
    \frac{dy}{dt} &= (1 - y)\phi(v_{R2}) - y\phi(v_{R1}) + \sum_{i=1}^3 c_{i2}(x, y)\xi_i(t),
\end{align*}
\]

(5)

where \( c_{i1}(x, y) \) and \( c_{i2}(x, y) \) are diffusion terms, and the three \( \xi_i(t) \) are uncorrelated Gaussian white noise with unit variances (see details in EC.5.1). Since the diffusion terms vanish at a rate of \( 1/\sqrt{n} \) as \( n \rightarrow \infty \), the evolutions of \( x \) and \( y \) could be approximated by the following ODE (Traulsen et al, 2005; Ohtsuki and Nowak, 2006)

\[
\begin{align*}
    \frac{dx}{dt} &= (1 - x)\phi(v_{C2}) - x\phi(v_{C1}), \\
    \frac{dy}{dt} &= (1 - y)\phi(v_{R2}) - y\phi(v_{R1}).
\end{align*}
\]

(6)

It is interesting to notice that ODE (6) is independent of \( f_C \), and only decided by the homophily indices \( h_C \) and \( h_R \), because this is true for (3). We remind the reader that \( f_C \) is almost surely independent of \( h_C \) and \( h_R \) (Proposition 2).

Before proceeding to the analysis of ODE (6), we provide some simple yet useful observations.
**Observation 1.** The following statements for $\phi(v_{C1})$ and $\phi(v_{C2})$ are true.

(a) At least one of $\phi(v_{C1})$ and $\phi(v_{C2})$ is zero;

(b) $\phi(v_{C1}) = \phi(v_{C2}) = 0$ if and only if $h_{C}x + (1 - h_{C})y = 1/2$.

Proof to the above observation is trivial. The reader only needs to recall the definition of $\phi(\cdot)$ and notice the relation $v_{C1} = -v_{C2}$. The observation below is analogous.

**Observation 2.** The following statements for $\phi(v_{R1})$ and $\phi(v_{R2})$ are true.

(a) At least one of $\phi(v_{R1})$ and $\phi(v_{R2})$ is zero;

(b) $\phi(v_{R1}) = \phi(v_{R2}) = 0$ if and only if $(1 - h_{R})x + h_{R}y = 1/2$.

### 6.3. ODE Analysis: Benchmark Cases

If the population consists entirely of conformists, i.e. $f_{C} = 1$, then the fashion game is equivalent to the network coordination game, and ODE (6) boils down to

$$\frac{dx}{dt} = (1 - x)\phi(v_{C2}) - x\phi(v_{C1}).$$

(7)

Notice again that $f_{C} = 1$ implies $h_{C} = 1$, so we have

$$v_{C1} = -v_{C2} = 2x - 1.$$  

(8)

Let’s calculate the fixed points of ODE (7). Suppose now $x$ is an interior fixed point, i.e. $x \notin \{0, 1\}$. Then Observation 1(a) tells us that it must be the case that both $v_{C1}$ and $v_{C2}$ are zero, because otherwise $(1 - x)\phi(v_{C2}) - x\phi(v_{C1})$ cannot be zero. From (8) we get immediately that $x = 1/2$. Notice also that $\frac{dx}{dt} > 0$ when $x > 1/2$ and $\frac{dx}{dt} < 0$ when $x < 1/2$, hence $1/2$ is unstable. It can be easily checked that $x = 0$ and $x = 1$ are both boundary fixed points, and they are locally asymptotically stable. Below is a summary of the above analysis.

**Proposition 8.** ODE (7) has three fixed points, $x = 0$, $x = 1$ and $x = 1/2$. The two boundary fixed points are locally asymptotically stable and the interior one is unstable.

In contrast, if the population consists of only rebels, i.e. $f_{C} = 0$, then the fashion game is equivalent to the network anti-coordination game and ODE (6) becomes

$$\frac{dy}{dt} = (1 - y)\phi(v_{R2}) - y\phi(v_{R1}).$$

(9)

In this case we have $v_{R1} = -v_{R2} = 1 - 2y$. And $1/2$ is still an interior fixed point. The difference is that it becomes globally stable, because $\frac{dy}{dt} < 0$ when $y > 1/2$ and $\frac{dy}{dt} > 0$ when $y < 1/2$. Also, it can be checked that neither 0 nor 1 is a fixed point in this case. To put it formally, we have the following proposition.

**Proposition 9.** ODE (9) has a unique fixed point, $y = 1/2$, which is globally stable.
6.4. ODE Analysis: The General Case

From Observations 1 and 2, we know that \((x, y) \in [0,1]^2\) is a fixed point of ODE (6) if and only if

\[
xφ(v_{C1}) = (1 - x)φ(v_{C2}) = yφ(v_{R1}) = (1 - y)φ(v_{R2}) = 0.
\]

(10)

We first calculate the interior fixed point \((x, y) \in (0,1)^2\), which, due to (10), must satisfy

\[
φ(v_{C2}) = φ(v_{C1}) = φ(v_{R2}) = φ(v_{R1}) = 0.
\]

Using Observation 1(b) and Observation 2(b), we get

\[
\begin{align*}
\{ & h_C x + (1 - h_C) y = 1/2, \\
& (1 - h_R)x + h_R y = 1/2, \}
\end{align*}
\]

(11)

The uniqueness of the interior fixed point depends crucially on the average homophily index, \(h = (h_C + h_R)/2\). If \(h \neq 1/2\), (11) has a unique solution \((1/2,1/2)\), which is independent of the homophily indices. However, if \(h = 1/2\), the two equations in (11) are linearly dependent and (11) has a line of solutions

\[
y = \frac{1}{2(1 - h_C)} - \frac{h_C}{1 - h_C} x.
\]

We now look at the boundary fixed points. Due to (10), \((0, y)\) is a fixed point if and only if

\[
\begin{align*}
\{ & φ(v_{C2}) = 0, \\
& (1 - y)φ(v_{R2}) = yφ(v_{R1}), \}
\end{align*}
\]

(12)

The first equation in (12) is equivalent to \(y \leq \frac{1}{2(1 - h_C)}\) and the second equation always has a unique solution: If \(h_R \geq 1/2\), the solution is \(y = \frac{1}{2h_R}\), and if \(h_R < 1/2\), the solution is \(y = 1\). Therefore, at \(x = 0\), it can be checked easily that (10) has no solution if \(h < 1/2\) and \(h_C < 1/2\), one solution \((0, \frac{1}{2h_R})\) if \(h \geq 1/2\) and \(h_R \geq 1/2\), and one solution \((0,1)\) if \(h_C \geq 1/2\) and \(h_R < 1/2\). Similarly, at boundary \(x = 1\), (10) has no solution if \(h < 1/2\) and \(h_C < 1/2\), one solution \((1, 1 - \frac{1}{2h_R})\) if \(h \geq 1/2\) and \(h_R \geq 1/2\), and one solution \((1,0)\) if \(h_C \geq 1/2\) and \(h_R < 1/2\).

On the other hand, at boundary \(y = 0\), \((x,0)\) is a fixed point if and only if

\[
\begin{align*}
\{ & (1 - x)φ(v_{C2}) = xφ(v_{C1}), \\
& φ(v_{R2}) = 0, \}
\end{align*}
\]

(13)

The second equation of (13) is equivalent to \(x \geq \frac{1}{2(1 - h_R)}\) and the first equation always has a unique solution: If \(h_C \geq 1/2\), the solution is \(x = \frac{1}{2h_C}\), and if \(h_C < 1/2\), the solution is \(x = 0\). Therefore, at \(y = 0\), (10) has one solution \((\frac{1}{2h_C}, 0)\) if \(h_C \geq 1/2\) and \(h \leq 1/2\), and no solution otherwise. Similarly, at boundary \(y = 1\), (10) has one solution \((1 - \frac{1}{2h_C}, 1)\) if \(h_C \geq 1/2\) and \(h \leq 1/2\), and no solution otherwise.

Local stabilities of these fixed points can be analyzed in the standard way, i.e. through computing the eigenvalues of the Jacobian matrices of ODE (6) (see details in EC.5.2). Fixed points and their stabilities are summarized in Table 1 and Theorem 4. It is notable that stability conditions are not affected by the switching probability \(φ\), but only decided by the homophily indices.
Table 1  Fixed points of ODE (6) and their stabilities.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Fixed points</th>
<th>Stability conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) ( h &lt; 1/2, h_C &lt; 1/2 )</td>
<td>((1/2, 1/2))</td>
<td>(h_R - h_C &gt; 0)</td>
</tr>
<tr>
<td>(ii) ( h &gt; 1/2, h_R \geq 1/2 )</td>
<td>((0, 1), (1, 1))</td>
<td>unstable</td>
</tr>
<tr>
<td>(iii) ( h &gt; 1/2, h_R &lt; 1/2 )</td>
<td>((1/2, 1/2), (1, 0))</td>
<td>stable</td>
</tr>
<tr>
<td>(iv) ( h &lt; 1/2, h_C &gt; 1/2 )</td>
<td>((0, 1), (1, 0))</td>
<td>unstable</td>
</tr>
<tr>
<td>(v) ( h_C = 1/2, h_R &lt; 1/2 )</td>
<td>((1/2, 1/2), (0, 1))</td>
<td>unstable</td>
</tr>
<tr>
<td>(vi) ( h = 1/2 )</td>
<td>(y = \frac{1}{2(1-h_R)} - \frac{h_C}{1-h_R} )</td>
<td>neutral stable</td>
</tr>
</tbody>
</table>

**Theorem 4.** The following statements about fixed points of ODE (6) are true.

(a) The interior fixed point \((1/2, 1/2)\) is stable if and only if \(h < 1/2\) and \(h_C < h_R\).

(b) If \(h > 1/2\), then boundary fixed points must be stable.

It is interesting to observe that Case (ii) in Table 1 is also consistent with half of Theorem 3. Theorem 3 states that for fashion games that are strong conformist homophilic, there exists a PNE such that all the conformists take the same action but different rebels may adopt different actions. This is echoed by the fact that in this case the ODE has boundary and non-corner fixed points (namely \((0, \frac{1}{2h_R})\) and \((1, 1 - \frac{1}{2h_C})\)). What’s more, the two fixed points are both stable.

In addition to local stabilities, structure of the trajectories of ODE (6) can be characterized by zero-isoclines analysis (see details in EC.5.3). Global dynamic behaviors of ODE (6) are presented in Theorem 5. The main result is that homophily, in general, inhibits the emergence of fashion cycles.

**Theorem 5.** The following statements about trajectories of ODE (6) are true.

(a) If fashion cycles, i.e. periodic solutions of ODE (6), emerge, then \(h < 1/2\) and \(h_C > h_R\).

(b) If \(h > 1/2\), then almost all trajectories of ODE (6) converge to boundary fixed points.

The only exception in Theorem 5(b) is the trajectory with initial point \((1/2, 1/2)\), because \((1/2, 1/2)\) is always a fixed point. The theorem above tell us that heterophily on average is a necessary condition for the emergence of fashion cycles. The main results of Theorem 4 and Theorem 5 are also illustrated in Figure 3.

The corollary below is actually a restatement of Theorem 4 and Theorem 5 (recall Proposition 1 and Proposition 3), describing the competing results of the three types of edges.

**Corollary 1.** The following statements about fixed points and trajectories of ODE (6) are true.

(a) The interior fixed point \((1/2, 1/2)\) is stable if and only if \(k_{CR} > k_{CC} + k_{RR}\) and \(k_{RR} > k_{CC}\).

(b) If \(k_{CR} < k_{CC} + k_{RR}\), then boundary fixed points are stable and almost all trajectories of ODE
Figure 3  Fixed points of ODE (6). Fashion cycles emerge only in the gray region. The interior fixed point \((1/2, 1/2)\) is stable in and only in the spokes region. Boundary fixed points are stable in the white region.

(6) converge to boundary fixed points.

(c) If ODE (6) has a periodic solution, then \(k_{CR} > k_{CC} + k_{RR}\) and \(k_{CC} > k_{RR}\).

Before providing more interpretations for the above corollary, let’s elaborate the stylized roles of CC and RR edges in the evolution of fashion. According to Proposition 8 and Proposition 9, it is quite intuitive that RR edges tend to push the trajectories of ODE (6) to the center while CC edges are likely to drive the trajectories to boundaries. The boundary-driving role of CC edges is also supported by the fact that the network coordination game has an exact potential function, i.e. a half of the total utility of all players. So starting from any action profile, the total utility of all players keeps increasing in SBRD. During this process, the action profile becomes nearer and nearer to one of the two extreme PNEs (one where all players choose 1 and the other where all choose 0), and the corresponding population state \(x\) approaches to one of the two boundaries. For the network anti-coordination game, a similar (but a little bit less convincing) argument suggests that the population state \(x\) approaches to 1/2 during SBRD. We remind the reader that in this case a half of the total utility of all players is still the exact potential function, and the action profile that maximizes this function corresponds to a population state that is near 1/2. Interestingly, supports for the center-driving role of RR edges can also be found in a voting model where “contrarians” lead to “hung-elections” (Galam, 2004), and a continuous social learning model where “rebels lead to the doctrine of the mean” (Cao et al., 2011). This is also related with a famous conjecture in graph theory, which claims, in terms of this paper, that each network anti-coordination game has a near-equilibrium (in the sense that each agent has a utility of at least -1) and the difference between the numbers of agents choosing 0 and those choosing 1 is at most 1. Thus in this near-equilibrium the population state \(y\) is approximately 1/2 (Bollobás and Scott, 2002, Conjecture 8).
The second condition in Corollary 1(a) now becomes quite intuitive: the interior fixed point \((1/2, 1/2)\) is stable only if RR edges are more powerful than CC edges. If the game consists of only CR edges, Theorem 1 claims that PNE does not exist almost always. So CR edges, roughly speaking, are inclined to make the system oscillate. This intuition is echoed by Corollary 1(b): if CR edges are not plenty enough, the system almost surely converges to a stable state and the existence of periodic solution is impossible. Yet, having plenty of CR edges alone is not enough to guarantee the emergence of fashion cycles, because if RR edges are more powerful than CC edges, the system may still be driven to the middle (Corollary 1(a)). Therefore, the emergence of fashion cycles relies on at least two conditions: plenty of CR edges, which lead to oscillation, and plenty of CC edges, which push the oscillation away from the interior fixed point \((1/2, 1/2)\). This is what said by Corollary 1(c).

Furthermore, the rotation direction of fashion cycles can be obtained by analyzing the zero-isoclines of ODE (6) (see also EC.5.3 for details).

**Corollary 2.** If fashion cycle exists, then the rotation of fashion cycle must be clockwise.

The above proposition is also intuitive. In a fashion cycle, starting from the situation that the majority of people using the same action, rebels first begin to switch to the unpopular action since they want to be distinctive. However, when conformists realize that the action of their (rebel) neighbors are different from them, they will change accordingly, which will lead again to the situation that most players take the same action (see also illustrations in Figure 4).

Finally, for regular graphs, homophily on average (i.e., \(h > 1/2\)) implies that the fashion game is relative homophilic (i.e., \(h_C > f_C\) and \(h_R > f_R\)). Therefore, an even clearer connection between the stabilities of fixed points and relative homophilies can be established. See Table EC.1 and Figure EC.1 for details.

### 6.5. How Good Is the Approximation?

Since the deriving of ODE (6) applies approximation techniques, the fixed points of ODE (6) may not correspond precisely to the PNEs (see Figure EC.4 for the examples), which do not precisely correspond to absorbing states of SBRD (see the example in Figure EC.3). Despite of this, numerical simulations on their trajectories suggest that short run behavior of the SBRD could be nicely approximated by ODE (6) (see an illustration in Figure 4). Furthermore, fixed points comparison demonstrates that the errors are fairly small, and they drop down quickly as the population size increases (see Figure 5, and EC.6.3 for details). In summary, our second approach is rather reliable.
7. Marketing Implications

Suppose action 0 and action 1 are two products, with identical functions and similar qualities but different brands, competing in the market, then \( z = xf_C + y(1 - f_C) \) can be naturally interpreted as the market share of the first product, and ODE (6) characterizes approximately the evolution of \( z \). This evolution can be taken as the “natural” evolution of market shares of the two products. To be precise, as long as the initial market shares are given by setting \( x \) and \( y \), everything subsequent will be determined by the choices of the consumers, whose naive behaviors are described by the best response dynamics.

Let’s do some comparisons between the two benchmark cases. Our results in Propositions 8 and 9 indicate that the competition in a market with all conformists may be much more fierce than that
in a market with all rebels. In the first market with all conformists, theoretically, there can be only one surviving product. When one product, say 0, loses a little bit to the other product (i.e. \( x < 0.5 \), the current market share of product 0 is less than 50%), then following the “natural trend”, her market share will eventually shrink to 0 (\( x = 0 \)). And the more she loses, the more difficult for her to reverse. There is a “Mattew Effect” or “positive feedback” behind the dynamics, which is caused by the herd behavior of consumers (who are all conformists). To avoid the potential complete failure, the first producer may decide to do more effort at the very beginning, say deeper price-cutting and more intensive advertising and marketing. Being clear of the situation too, the second producer may do the same thing to keep her advantage, and thus the competition can be really cut-throat.

For the market with all rebels, however, things are opposite. There is inherently an “anti-Mattew Effect” or “negative feedback”, which is caused by the anti-herd behavior of consumers (who are all rebels). If the first producer is on the inferior position at the beginning (i.e. \( y < 0.5 \)), then she does not need to be worried at all, because the “natural trend” is on her side. Being clear of the situation too, the second producer has no incentive to enhance marketing or advertising, or to cut price, since the more advantage she has in the market, the more difficult it is for her to obtain even bigger market share. Eventually, it is likely that both products survive and their market shares are half-half (\( y = 0.5 \)).

The above logic is hard to test directly, because practically it is quite challenging to say whether a consumer is a conformist or a rebel. However, a fashion game with all conformists can be taken as a market for selling products with positive network externality (say office softwares, online e-games, and social network services) and that with all rebels is mathematically equivalent to a market for selling products with network externality (say luxury goods and fashionable clothing). Evidences can be found easily to support the proposition that competition in the first kind of markets is usually much more fierce than in the second one. For instance, some fashionable goods companies

---

**Figure 5** Error analysis: fixed-point differences between ODE and SBRD. (a) \( n = 100, f_C = 0.5 \). (b) \( n = 400, f_C = 0.5 \). See EC.6 for simulation details.
even do some “anti-marketing” behavior in the sense that they do not advertise but try to hide the information of their products (Yoganarasimhan, 2012a).

Our analysis may also indicate that for a market selling products with complements, new entrants have little chance to get any share. For a market selling products with substitutes, however, new entrants always have chances, in fact, even advantages. This may shed some new light on the anti-monopoly problem.

8. Conclusions

In this paper, we provide a new structural factor for the emergence of fashion cycles, suggesting that homophily, in general, is on the negative side of this interesting phenomenon. Our result is intuitively plausible. Remember that there are three elementary two-player games in the fashion game, the coordination game, the anti-coordination game, and Matching Pennies. Pure Nash equilibrium exists in the former two games but not in the third one. So in order to observe a “global oscillation”, there should be enough conformist-rebel interactions, which naturally corresponds to a network with heterophily.

While the mathematical model we use is to attack the phenomenon of fashion, and the terms in this paper are all about fashion, it may be actually interpreted under other scenarios. In a paper in memory of Simmel and his study on fashion, Benvenuto (2000) gives twelve other pairs of two opposite tendencies that are similar to imitation and distinction, e.g., universality and particularity, submission and sense of power, femininity and masculinity, stillness and movement, creation and destruction, Darwinian Selection and Darwinian Mutation, etc. Several more pairs can be thought out immediately, say obedience and rebellion, imitation and innovation. More technically, there are also several natural pairs of terms, say positive feedback and negative feedback, coordination and anti-coordination, strategic complements and strategic substitutes, supermodular and submodular. So research on how the two competing forces interact with each other may have some universal implications.

As to motivations, why people want to be like or to be different with the others goes also far beyond fashion. Young (2001) summarizes three kinds of reasons why people want to be like the others. The first, which he names imitative conformity or pure conformity, is exactly one side of fashion. People want to be like the others simply because otherwise they don’t feel good. The second, which he named instrumental conformity, facilitates people’s livings. Just as argued by Young, “people drive on the same side of road not because it is fashionable, but because they want to avoid collisions.” The forming of many conventions falls into this logic (Berninghaus and Schwalbe, 1996; Young, 2001). The last, which he names informational conformity, says that “people adopt
a behavior they observe around them because the demonstration of effects”, i.e. they get useful information telling them that that particular behavior is good.

As to why people want to be different with the others, we did not find a similar classification in literature. Nevertheless, the analogous taxonomy is almost immediate: pure deviation, instrumental deviation, and informational deviation. People may think that being different is fashionable, being different may give them a great convenience in living (say go shopping on Monday morning, choose a road to drive home that few people notice, etc.), and observing that most people around her doing the same thing may tell her a very important piece of information which makes her choose being different. For instance, when most people around you are buying stock, it is very likely that there is a great bubble in the stock market and it is better for you to sell as soon as possible.

Also, actions in this paper can be safely taken as people’s opinions, so our research is closely related with the well studied field of (non-Bayesian) social learning. In this field, BRD is also frequently used. Hence we hope that our research might enrich this field. See the conclusion section of Cao and Qin et al. (2013) for more perspectives on this model.

Acknowledgments
This work is supported by the 973 Program (2010CB731405) and National Natural Science Foundation of China (71101140). We gratefully acknowledge helpful discussions with Xujin Chen, Zhiwei Cui, Josef Hofbauer, Xiaodong Hu, Weidong Ma, Karl Sigmund, Arne Traulsen, Jin Xu, Changjun Wang, Lin Zhao, Wei Zhu, participants of The 2012 Fall Workshop: Topics in Game Theory and Applications, Shandong University, and seminar participants in Qufu Normal University, Shandong University, Zhengzhou University, University of Vienna, and Max-Planck-Institute for Evolutionary Biology.

References


[50] Yoganarasimhan H (2012b) Identifying the Presence and Cause of Fashion Cycles in the Choice of Given Names, UC Davis working paper.
Electronic Companion

EC.1. Proofs in Section 3

EC.1.1. Proof to Proposition 2

Proposition 2. (Restated) For any three rational numbers \( f_C \in (0, 1), h_C, h_R \in [0, 1) \), there exists a fashion game with conformist fraction \( f_C \), conformist homophily index \( h_C \), and rebel homophily index \( h_R \).

Proof. When at least one of \( h_C \) and \( h_R \) is zero, arguments in Subsection 3.2 can show that the proposition is valid, so we assume in the rest of this proof that \( h_C \) and \( h_R \) are both positive. We use a parameter \( \alpha \) to tune \( f_C \), and the other parameter \( \beta \) to tune \( h_C \) and \( h_R \).

Let the fashion game \( G \) have \( \alpha \) conformists and \( 1 - f_C \)\( \alpha \) rebels, i.e. \( c(G) = \alpha \) and \( r(G) = \frac{1 - f_C}{f_C} \alpha \). We also let \( k_{CC}(G) = \beta, k_{RR}(G) = \frac{2(1 - h_C)}{h_C} \beta, \) and \( k_{CR}(G) = \frac{h_R(1 - h_C)}{(1 - h_R)h_C} \beta \). Then it can be checked easily that if such a fashion game exists, then all the three indices are satisfied.

We prove that such a fashion game does exist. First of all, we can make sure that all the above numbers are integer (by taking large enough \( \alpha \)). Three more things may go wrong: (i) even when all pairs of nodes within conformists are connected, the demand for \( k_{CC}(G) \) CC edges still cannot be met, i.e. \( \beta > \left( \frac{c(G)}{2} \right) \); (ii) similarly, the demand for \( k_{RR}(G) \) RR edges may exceed the upper bound \( \left( \frac{r(G)}{2} \right) \), and (iii) \( k_{CR}(G) \) may exceed the upper bound \( c(G)r(G) \). However, it can be seen that as long as \( \beta \) is moderately small w.r.t. \( \alpha \), none of the three worries is a problem. Q.E.D.

EC.1.2. Proof to Proposition 3

Proposition 3. (Restated) A fashion game \( G \) is homophilic on average if and only if \( k_{CR}(G) < k_{CC}(G) + k_{RR}(G) \), and heterophilic on average if and only if \( k_{CR}(G) > k_{CC}(G) + k_{RR}(G) \).

Proof. We only prove the first part. For given \( G, h(G) > 1/2 \) is equivalent to

\[
\frac{2k_{CC}(G)}{2k_{CC}(G) + k_{CR}(G)} + \frac{2k_{RR}(G)}{2k_{RR}(G) + k_{CR}(G)} > 1
\]

\[
\iff 4k_{CC}(G)k_{RR}(G) > k_{CR}(G)^2
\]

\[
\iff (k_{CC}(G) + k_{RR}(G))^2 - (k_{CC}(G) - k_{RR}(G))^2 > k_{CR}(G)^2
\]

\[
\iff k_{CC}(G) + k_{RR}(G) > k_{CR}(G).
\]

Hence the proposition. Q.E.D.
EC.1.3. Proof to Proposition 5

**Proposition 5 (Restated)** Let \( G \) be a fashion game. If the underlying graph of \( G \) is regular, then the three concepts, conformist relative homophily, rebel relative homophily and homophily on average, are equivalent.

**Proof.** We shall prove that in regular graphs, the first two concepts are both equivalent to

\[
    h_C + h_R > 1, \quad \text{(EC.1)}
\]

which is exactly the definition of the third concept.

It can be checked that

\[
    f_C(1 - h_C) = f_R(1 - h_R). \quad \text{(EC.2)}
\]

Therefore formula (EC.1) is equivalent to

\[
    h_C > \frac{f_C}{1 - f_C} (1 - h_C)
\]

\[
    \iff \frac{h_C}{1 - h_C} > \frac{f_C}{1 - f_C}
\]

\[
    \iff h_C > f_C
\]

\[
    \iff h_R > f_R,
\]

where the second last equivalence is from the fact the \( \frac{x}{1-x} \) is an increasing function of \( x \) in \([0,1)\), and the last equivalence is due to the symmetric relation between conformists and rebels in (EC.1) and (EC.2). Q.E.D.

EC.2. Proofs in Section 4

EC.2.1. Proof to Theorem 1

**Theorem 1 (Restated)** Let \( G \) be a fashion game. If \( G \) satisfies complete heterophily, i.e. \( h(G) = 0 \), and at least one agent has an odd degree, then \( G \) does not possess any PNE.

**Proof.** \( h(G) = 0 \) implies that all edges are inter-type ones, and there is no inner-type edge. Let \( ij \) be an arbitrary such edge, it must contribute 1 to the utility of one player and -1 to the other player, no matter which pure action profile is chosen. Therefore, for any action profile (pure or mixed), the total utility of all players is always 0. For any action profile to be a Nash equilibrium, it must hold that every player has a utility of 0, because otherwise (i.e. one player has a strictly positive utility), there would exist one player having a negative utility, contradicting the hypothesis of Nash equilibrium. However, this is absolutely impossible for the odd degree player. Q.E.D.
EC.2.2. Proof to Theorem 2

THEOREM 2 (Restated) Let $G$ be a fashion game. If $G$ satisfies complete homophily, i.e. $h(G) = 1$, then $G$ is an exact potential game, and thus possesses at least one PNE.

Proof. We shall prove that the potential function can be simply defined as a half of the total utility of all players, i.e.

$$p(c) = 0.5 \sum_{i \in N} u_i(c), \forall c \in \{0, 1\}^N.$$  \hspace{1cm} (EC.3)

Since $h(G) = 1$, we know that all edges are inner-type, and hence for any action profile $c \in \{0, 1\}^N$, and any edge $ij \in E$, $i$ is liked by $j$ if and only if $j$ is liked by $i$, i.e.

$$i \in L_i(c, G) \Leftrightarrow j \in L_j(c, G), \forall c \in \{0, 1\}^N.$$  \hspace{1cm} (EC.4)

(EC.4) indicates that

$$|L_i(c, G)| = |j \in N_i(G) : i \in L_j(c, G)|, \forall c \in \{0, 1\}^N.$$  \hspace{1cm} (EC.5)

$$|D_i(c, G)| = |j \in N_i(G) : i \in D_j(c, G)|, \forall c \in \{0, 1\}^N.$$  \hspace{1cm} (EC.6)

Let $a \in \{0, 1\}^N$ be an arbitrary pure action profile. Suppose it is not a PNE, and $i$ is unsatisfied with $a$, then

$$u_i(a) = |L_i(a, G)| - |D_i(a, G)| < 0.$$  \hspace{1cm} (EC.7)

Suppose w.l.o.g. that $a_i = 0$, and let $b = (1, a_{-i})$ be the pure action profile derived by letting $i$ switch her action from 0 to 1. Then

$$L_i(a, G) = D_i(b, G), D_i(a, G) = L_i(b, G).$$  \hspace{1cm} (EC.8)

Combining (EC.7) and (EC.8), we have

$$u_i(b) - u_i(a) = 2(|D_i(a, G)| - |L_i(a, G)|).$$

(EC.5)(EC.6)(EC.8) imply that

$$\{j \in N_i(G) : u_j(b) > u_j(a)\} = D_i(a, G),$$

$$\{j \in N_i(G) : u_j(b) < u_j(a)\} = L_i(a, G).$$

Because there is only one player, $i$, who switches action from $a$ to $b$, we know that each of $i$’s “enemy” in $D_i(a, G)$ gets a benefit of 2, and each of her “friend” in $L_i(a, G)$ gets a harm of 2. To be precise,

$$u_j(b) - u_j(a) = 2, \forall j \in D_i(a, G),$$
By definition of (EC.3) and the above arguments, we finally have

\[ u_j(b) - u_j(a) = -2, \forall j \in L_i(a, G). \]

This completes the whole proof. Q.E.D.

**EC.3. Proof to Proposition 7**

**Proposition 7 (Restated)** Let $G$ be a fashion game.

(a) If $G$ satisfies complete homophily, i.e. $h(G) = 1$, then it satisfies both strong conformist homophily and strong rebel homophily.

(b) If $G$ satisfies both strong conformist homophily and strong rebel homophily, then $h(G) \geq 0.5$.

**Proof.** (a) Strong conformist homophily is obvious because $h(G) = 1$ implies that $h_i(G) = 1$ for all players. To show that strong rebel homophily is also true, let $a$ be an arbitrary PNE of $G$. Due to Theorem 2, $a$ exists. We define $R_1$ as the set of rebels that choose 0 in $a$, and $R_2$ the set of rebels choosing 1 in $a$. That $a$ is a PNE tells us that all the rebels are satisfied in $a$, i.e. each player $i \in R$ likes at least a half of her neighbors. Note that $h(G) = 1$ means rebels only have rebel neighbors, so strong rebel homophily holds.

(b) If a fashion game satisfies strong conformist homophily, then $h_C \geq 0.5$, because $h_C$ is a convex combination for all the $h_i(G), i \in C$. Similarly, since for any strong rebel homophily fashion game $G$, it holds for all $i \in R$ that $h_i(G) \geq 0.5$, it must be true that $h_R \geq 0.5$. Thus, $h_C + h_R \geq 1$. Q.E.D.


To streamline the analysis, we shall present three concepts that are relaxations of PNE, *weak PNE component*, *PNE dominating set* and *strong PNE component*.
EC.4.1. PNE Components and PNE Dominating Set

First of all, we would like to remind the reader three elementary concepts from the graph theory. Given a graph \( g = (N, E) \), we say that a subset of nodes \( S \) is an independent set if no pair of nodes in \( S \) are linked to each other, i.e. \( ij \notin E, \forall i, j \in S \). We say that \( S \) is a dominating set of \( g \) if each node in \( N \setminus S \) has at least one neighbor in \( S \), i.e. \( N_i(g) \cap S \neq \emptyset, \forall i \in N \setminus S \). Let \( E(S) \) be the inner edges of \( S \), i.e. \( E(S) = \{ ij : i, j \in S, ij \in E \} \). Then \( g|_S = (S, E(S)) \) is called the induced subgraph of \( g \) derived by \( S \). The above terms are also abused to apply to network games.

When PNE is not guaranteed (as in the case of the fashion game), it is natural to ask how well we can we “approximate” PNE. By approximate, we mean a pure action profile such that a large proportion of players are satisfied (i.e. have no incentive to deviate). If this proportion is 1, then we get exactly a PNE. Thus the largest possible such proportion is a natural measure of PNE approximation, and the associated pure action profile is an extension of PNE.

In the following, we use \( \Lambda = (N,(A_i)_{i \in N},(u_i)_{i \in N}) \) to denote a general normal form game, where \( N \) is the set of players, \( A_i \) the action set of player \( i \), and \( u_i \) the utility function of \( i \). For any player set \( S \subseteq N \), we also let \( A_S = \prod_{i \in S} A_i \) to denote the set of action profiles of \( S \). In the following, when we say “a network game”, we mean a strategic form game associated with a network, where the utility of each player is only decided by the actions of her neighbors as well as that of herself.

**Definition EC.1 (Weak PNE Component).** Let \( \Lambda = (N,(A_i)_{i \in N},(u_i)_{i \in N}) \) be a normal form game, and \( S \subseteq N \) a subset of players. Suppose \( a_S \in A_S \) is a pure action profile of \( S \). If there exists \( b_{N\setminus S} \in A_{N\setminus S} \) such that no player in \( S \) has incentive to deviate in the action profile \((a_S,b_{N\setminus S})\), i.e.

\[
    u_i(c_i,a_{S\setminus\{i\}},b_{N\setminus S}) \leq u_i(a_i,a_{S\setminus\{i\}},b_{N\setminus S}), \forall i \in S, \forall c_i \in A_i \setminus \{a_i\},
\]

we call \( a_S \) a weak PNE component (w.r.t. \( b_{N\setminus S} \)).

When there exists \( a_S \in A_S \) such that \( a_S \) is a weak PNE component, we say that \( S \) possesses a weak PNE component. A weak PNE component \( a_S \) is called maximal if there is no \( j \in N \setminus S \) and \( b_j \in A_j \) such that \((a_S,b_j)\) is also a weak PNE component.

**Proposition EC.1.** Weak PNE components have the following properties.

(a) Suppose \( \Lambda \) is a normal form game. If \( a_S \) is a maximal weak PNE component w.r.t. \( b_{N\setminus S} \), then no player in \( N \setminus S \) is satisfied with the pure action profile \((a_S,b_{N\setminus S})\).

(b) Suppose \( G \) is a network game. If \( S \) is an independent set, then \( S \) possesses a weak PNE component.

(c) Suppose \( G \) is a network game. Let \( N_S(G) \) be the union of neighbor sets of players in \( S \), i.e.

\[
    N_S(G) = \bigcup_{i \in S} N_i(G),
\]
and $I$ an independent set of the induced subgraph $G|_{N \setminus (S \cup N_S(G))}$. If $S$ possesses a weak PNE component, then so does $S \cup I$.

(d) Suppose $G$ is a network game. If $S$ possesses a maximal weak PNE component, then $S$ is a dominating set of $G$.

Proof. Part (a) is obvious, because otherwise the hypothesis that $a_S$ is maximal would be violated.

We shall prove a result stronger than (b). Suppose now $S$ is an independent set, and let $b_{N \setminus S}$ be an arbitrary pure action profile of $N \setminus S$. Fix $b_{N \setminus S}$, we let players in $S$ choose actions they like most one by one (in an arbitrary order). Note that in network games, the payoff of any player is affected only by the actions of her neighbors (and the action of her own). Since no pair of players in $S$ are neighboring to each other, we know that the actions of later players do not affect the payoffs of the earlier ones. Therefore, all players in $S$ will eventually be satisfied, and hence $S$ possesses a weak PNE component w.r.t. $b_{N \setminus S}$.

To prove (c), suppose $a_S$ is a weak PNE component w.r.t. $b_{N \setminus S}$. Similar to the proof of (b), we fix $a_S$ and let the players in $I$ choose a best response one by one (in an arbitrary order). Let their choices be $c_I$. That $I$ is an independent set of $G|_{N \setminus (S \cup N_S(G))}$ indicates that none of the players in $S$ will be affected, and neither will the deviation of later players in $I$ affect the earlier ones. Thus $(a_S, c_I)$ is a weak PNE component w.r.t. $b_{N \setminus (S \cup I)}$.

(d) is an immediate corollary of (c), because $S$ is a dominating set of $G$ if and only if $N \setminus (S \cup N_S(G)) = \emptyset$. Q.E.D.

Proposition EC.2. Let $\Lambda = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ be a normal form game. If each player has at most $m$ actions, i.e. $|A_i| \leq m$, $\forall i \in N$, then there exists $S \subseteq N$, such that

$$|S| \geq n/m,$$

and $S$ possesses a weak PNE component.

Proof. We prove via a simple application of the probabilistic method (c.f. Alon and Spencer, 2007). Let each player choose a pure action uniformly randomly, then the probability that player $i$ is satisfied is at least $1/|A_i| \geq 1/m$. Therefore, the expected number of satisfied players is at least $n/m$, and hence there must exist a pure action profile where at least $n/m$ players are satisfied (because otherwise the expected number of satisfied players would be strictly less than $n/m$). This completes the proof. Q.E.D.

The above proposition tells us that when the pure action space of each player is small, then there must exist a weak PNE component with a nontrivial size.
DEFINITION EC.2 (STRONG PNE COMPONENT). Let \( \Lambda = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \) be a normal form game. \( \forall S \subseteq N, a_S \in A_S \) is called a strong PNE component of \( \Lambda \) if and only if for all \( a_{N \setminus S} \in A_{N \setminus S} \), no player in \( S \) has incentive to deviate in the action profile \( (a_S, a_{N \setminus S}) \).

It is easy to construct examples to show that Proposition EC.1 and Proposition EC.2, in general, do not hold for strong PNE component. However, it has a special property of its own.

PROPOSITION EC.3. Let \( \Lambda \) be a normal form game. If \( X \) and \( Y \) are two disjoint player sets, \( a_X \) and \( a_Y \) are two strong PNE components, then \( (a_X, a_Y) \) is also a strong PNE component.

**Proof.** Suppose \( c_{N \setminus (X \cup Y)} \in A_{N \setminus (X \cup Y)} \), then \( a_X \) is a strong PNE components means that all players in \( X \) are satisfied in the action profile \( (a_X, b_Y, c_{N \setminus (X \cup Y)}) \). For the same reason, players in \( Y \) have no incentive to deviate either. Hence the proposition. Q.E.D.

Note that the above property is not held, in general, by weak PNE component. To see this, it is enough to observe that any single set possesses a weak PNE component. If on the contrary the above property holds, then we would get that the whole set of \( N \), which is a union of single player sets, possesses a weak PNE too. This means we would derive that PNE exists for any strategic game, which is obviously impossible.

DEFINITION EC.3 (PNE DOMINATING SET). Let \( \Lambda = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \) be a normal form game and \( S \subseteq N \) a subset of players. We say \( S \) is a PNE dominating set if for each \( b_{N \setminus S} \in A_{N \setminus S} \), \( S \) possesses a weak PNE component w.r.t. \( b_{N \setminus S} \).

The definition of \( N_S(G) \) in Proposition EC.1 is still valid in the following proposition.

PROPOSITION EC.4. Let \( G \) be a network game.

(a) Any independent set of \( G \) is a PNE dominating set.

(b) If two disjoint player sets \( X \) and \( Y \) are both PNE dominating sets, no player in \( X \) has any neighbor in \( Y \), and no player in \( Y \) has any neighbor in \( X \), i.e.

\[
N_X(G) \cap Y = \emptyset, N_Y(G) \cap X = \emptyset,
\]

then \( X \cup Y \) is also a PNE dominating set.

(c) If \( S \) is a PNE dominating set, and \( I \) an independent set of the induced subgraph \( G|_{N \setminus (S \cup N_S(G))} \), then \( S \cup I \) is a PNE dominating set of \( G \).

(d) If \( S \) is a maximal PNE dominating set, then it must be a dominating set of the underlying graph.

**Proof.** (a) is an improvement of Proposition EC.1(b), and is already implied in the proof to it. (d) is a direct implication of Proposition EC.1(d). (c) is implied in the combination of (a) and (b). So we are left to show that (b) is true.
Conditions in (EC.9) tell us that the payoffs of players in \( X \) are affected only by players in \( N \setminus (X \cup Y) \) (and their own actions), but not by \( Y \), and the payoffs of \( Y \) are not affected by \( X \). Due to the above reason and the hypothesis that \( X \) and \( Y \) are both PNE dominating sets, for arbitrary action profile of \( N \setminus (X \cup Y) \), we can make all the players in \( X \cup Y \) be satisfied, and hence (b) is true. Q.E.D.

It is easy to show that conditions in (EC.9) are indispensable. The observations below can also easily shown to be true, (i) a PNE is the largest possible strong PNE component, (ii) when \( S \) possesses a strong PNE component, it must be a PNE dominating set, (iii) any PNE dominating set must possess a weak PNE component, (iv) when PNE exists, the sets of PNEs, largest strong PNE components and largest weak PNE components all coincide.

EC.4.2. Partial Potential Games

**Definition EC.4 (Partial Potential Games).** Let \( \Lambda = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \) be a normal form game (with at least three players). \( S \) is a subset of players and \( b_{N \setminus S} \in A_{N \setminus S} \) a pure action profile of \( N \setminus S \). \( \Lambda(b_{N \setminus S}) = (S, (A_i)_{i \in S}, (u_i(b_{N \setminus S}))_{i \in S}) \) is the reduced game of \( \Lambda \) derived from fixing the actions of players in \( N \setminus S \) as \( b_{N \setminus S} \), where

\[
u_i(b_{N \setminus S})(a_S) = u_i(a_S, b_{N \setminus S}), \forall i \in S, \forall a_S \in A_S.
\]

We say that \( \Lambda \) is a partial potential game w.r.t. \( b_{N \setminus S} \) if the reduced game \( \Lambda(b_{N \setminus S}) \) is a potential game.

We say that \( \Lambda \) is a partial potential game w.r.t. \( N \setminus S \) if it is a partial potential game w.r.t. all \( b_{N \setminus S} \in A_{N \setminus S} \).

We remark that in the above definition, \( S \) is allowed to be equal to the empty set and \( N \). And we assume that any “game” with one or none player is a potential game. So when \( |S| \leq 1 \), \( \Lambda \) is always a partial potential game w.r.t. \( N \setminus S \). Also, if \( \Lambda \) is a partial potential game w.r.t. \( \emptyset \), then it is a potential game.

The close relations between partial potential games and PNE components and PNE dominating sets are expressed in the following theorem.

**Theorem EC.1.** Let \( \Lambda = (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \) be a normal form game. \( S \) is a subset of players.

(a) If there exists \( b_{N \setminus S} \in A_{N \setminus S} \) such that \( \Lambda \) is a partial potential game w.r.t. \( b_{N \setminus S} \), then \( S \) possesses a weak PNE component.

(b) If \( \Lambda \) is a partial potential game w.r.t. \( N \setminus S \), then \( S \) is a PNE dominating set.

(c) If \( b_{N \setminus S} \in A_{N \setminus S} \) is a strong PNE component of \( \Lambda \), and \( \Lambda \) is a partial potential game w.r.t. \( b_{N \setminus S} \), then \( \Lambda \) possesses a PNE.
Proof. That $\Lambda$ is a partial potential game w.r.t. $b_{N \setminus S}$ indicates that the reduced game $\Lambda(b_{N \setminus S})$ possesses a PNE, which is a weak PNE component of $N \setminus S$. Hence (a) is true. Recall Definition EC.3 and Definition EC.4, (b) is an immediate implication of (a).

To prove (c), let $c_S$ be a PNE for the reduced game $\Lambda(b_{N \setminus S})$. Then $(c_S, b_{N \setminus S})$ is a PNE for $G$. Q.E.D.

EC.4.3. Application on the Fashion Game

For the fashion game, since each player has exactly 2 actions, we get immediately from Proposition EC.2 that the largest weak PNE component consists of at least a half of all the players. In fact, using the partial potential arguments introduced in the previous subsection, we have the following stronger and more concrete results.

**Theorem EC.2.** Suppose $G$ is a fashion game. The following statements are true.

(a) $G$ is a partial potential game w.r.t. $R$, and hence $C$ is a PNE dominating set.
(b) $G$ is a partial potential game w.r.t. $C$, and hence $R$ is a PNE dominating set.
(c) Let $H^*$ be the set of players that have an individual homophily index of 1, i.e.

$$H^* = \{j \in N : h_i(G) = 1\},$$

then $G$ is a partial potential game w.r.t. $N \setminus H^*$, and hence $H^*$ is a PNE dominating set.
(d) Let $X \in \{C, R, H^*\}$, and $I^*(X)$ be an independent set of the induced subgraph $G|_{N \setminus X}$, then $X \cup I^*(X)$ is also a PNE dominating set.

Proof. To prove (a)-(c), we need a simple property of potential games. Suppose $\Lambda = (N, (A_i)_{i \in N}, (u_i)_{i \in N})$ is a potential game, then $\Lambda' = (N, (A_i)_{i \in N}, (v_i)_{i \in N})$ is also a potential game, where there exists a constant $\delta_i$ such that

$$v_i(a) = u_i(a) + \delta_i, \forall i \in N, \forall a \in A_N.$$  

(EC.11)

This property is true because the preferences of all players in $\Lambda'$ are exactly the same as in $\Lambda$.

To prove (a), let $a_R$ be an arbitrary pure action profile of the rebel set $R$. We construct a new fashion game $\Lambda'(a_R)$ from the reduced game $\Lambda(a_R)$ by letting the payoff of each player in $\Lambda'(a_R)$ be the number of neighbors she likes minus that she hates. Then $\Lambda'(a_R)$ is a fashion game and there exists constant $\delta_i$ such that (EC.11) holds. Since $\Lambda'(a_R)$ is a fashion game with only conformists, we know from Theorem 2 that it is a potential game. Hence using the above property we know that $\Lambda(a_R)$ is also a potential game.

Similar to the proof to (a), (b) and (c) can be shown to be true using a combination of the above property and Theorem 2. (d) is a simple application of Proposition EC.4(d). Q.E.D.
PROPOSITION EC.5. Let $G$ be a fashion game.

(a) If $G$ is strong conformist homophilic, then $C$ possesses a strong PNE component.

(b) If $G$ is strong rebel homophilic, then $R$ possesses a strong PNE component.

Proof. (a) is true because we can simply let all the conformist choose 0. (b) is also easy, because we can let players in $R_1$ choose 0 and players in $R_2$ choose 1, where $R_1$ and $R_2$ are the same as in Definition 7. Q.E.D.

Combining Theorem EC.1(c), Theorem EC.2(a)(b), and Proposition EC.5, we have the main result of Theorem 3.

THEOREM 3 (RESTATED) Let $G$ be a fashion game. If $G$ satisfies strong conformist homophily or strong rebel homophily, then PNE exists.

EC.4.4. Bridging PNE Possessing Games and Potential Games

DEFINITION EC.5. For any normal form game $\Lambda$ with $n$ players, we use $f(\Lambda)$ to denote the largest number of “free” players, i.e.

$$f(\Lambda) = \max\{|S| : \exists \text{ strong PNE component } b_{N\setminus S} \text{ s.t. } \Lambda \text{ is a partial potential game w.r.t. } b_{N\setminus S}\},$$

and define its potential degree as

$$p(\Lambda) = \frac{f(\Lambda)}{n}.$$

When $\Lambda$ does not have any strong PNE component at all, we simply define $f(\Lambda)$ as $-\infty$. It is valuable to remark that any PNE possessing game has a nonnegative potential degree, because $S = \emptyset$ is allowed in the definition of $f(\Lambda)$ and any PNE is a strong PNE component. In fact, the propositions below are obvious.

PROPOSITION EC.6. $\Lambda$ has a PNE if and only if $p(\Lambda) \geq 0$.

PROPOSITION EC.7. $\Lambda$ is a potential game if and only if $p(\Lambda) = 1$.

Intuitively, games with larger potential degrees can reach PNEs through more flexible dynamics. In particular, games with potential degree zero are the most fragile ones, and potential games are the most flexible ones. Thus the concept of potential degree does bridge potential games and PNE possessing games.
EC.5. Details of Section 6

EC.5.1. Derivation of ODE (5)

We renumber the four sets of players $C^1$, $C^2$, $R^1$ and $R^2$ as groups 1, 2, 3 and 4, respectively, and thus conveniently write the transition probabilities as

$$
T_{12} = P(n_C^1 \rightarrow n_C^1 - 1), T_{11} = 1 - T_{12},
T_{21} = P(n_C^1 \rightarrow n_C^1 + 1), T_{22} = 1 - T_{21},
T_{34} = P(n_R^1 \rightarrow n_R^1 - 1), T_{33} = 1 - T_{34},
T_{43} = P(n_R^1 \rightarrow n_R^1 + 1), T_{44} = 1 - T_{43}.
$$

(EC.12)

Note that the underlying variable time $t$ is unwritten, and $T_{ij} = 0$ for all the other values of $i$ and $j$ that are not defined above.

Denote the population densities of the four groups of players (at time $t$) by $x \equiv (x_1, x_2, x_3, x_4) = (f_C x, f_C (1 - x), (1 - f_C) y, (1 - f_C) (1 - y))$.

Let $\rho(x)$ be the probability density that $x$ is reached, then a Fokker-Planck equation for $\rho(x)$ can be derived by applying Kramers-Moyal expansion (Traulsen et al., 2005; Traulsen et al., 2006)

$$
\frac{d\rho(x)}{dt} = -\sum_{k=1}^{3} \frac{\partial (\rho(x) a_k(x))}{\partial x_k} + \frac{1}{2} \sum_{j,k=1}^{3} \frac{\partial^2 (\rho(x) b_{jk}(x))}{\partial x_k \partial x_j},
$$

(EC.13)

where the drift terms are given by

$$
a_k(x) = \sum_{j=1}^{4} \left( T_{jk}(x) - T_{kj}(x) \right),
$$

and the diffusion matrix is given by

$$
b_{jk}(x) = \frac{1}{n} \left( -T_{jk}(x) - T_{kj}(x) + \delta_{jk} \sum_{l=1}^{4} \left( T_{jl}(x) + T_{lj}(x) \right) \right).
$$

Since noises arise only from stochastic updating and are therefore not correlated, the Itô calculus can be applied to derive a Langevin equation describing the time evolution of $x$ (Traulsen et al., 2006). Applying this approach, ODE (EC.13) corresponds to the stochastic replicator-mutator equations

$$
\frac{dx_k}{dt} = a_k(x) + \sum_{i=1}^{3} c_{ki}(x) \xi_i(t), k = 1, 2, 3,
$$

(EC.14)

where $c_{kj}$ is defined by $\sum_{i=1}^{4} c_{ki}(x) c_{ij}(x) = b_{kj}(x)$ and each element of the vector $\xi$ is a Gaussian white noise with unit variance. Therefore, $c_{ki}(x)$ vanishes at a rate of $\frac{1}{\sqrt{n}}$ as $n \to \infty$. From (EC.12), we have

$$
a_1(x) = T_{21}(x) - T_{12}(x),
a_2(x) = T_{12}(x) - T_{21}(x),
a_3(x) = T_{43}(x) - T_{34}(x).
$$
Notice that \( x_1 + x_2 = f_C, x_3 + x_4 = 1 - f_C \), ODE (EC.14) could be simplified as

\[
\begin{cases}
\frac{dx_1}{dt} = T_{21}(x) - T_{12}(x) + \sum_{i=1}^{3} c_{1i}(x)\xi_i(t), \\
\frac{dx_3}{dt} = T_{43}(x) - T_{34}(x) + \sum_{i=1}^{3} c_{3i}(x)\xi_i(t).
\end{cases} \tag{EC.15}
\]

Let \( c_{1i}(x,y) = f_C c_{1i}(x) \) and \( c_{3i}(x,y) = (1 - f_C) c_{3i}(x) \), and recall the definitions of (4) and (EC.12), it can be checked that the above ODE (EC.15) and ODE (5) are equivalent.

**EC.5.2. Proof to Table 1 and Theorem 4**

At the interior fixed point \((1,2,1,2)\), the Jacobian matrix of ODE (6) is given by

\[
\begin{pmatrix}
-2\phi'(0)h_C & -2\phi'(0)(1-h_C) \\
2\phi'(0)(1-h_R) & 2\phi'(0)h_R
\end{pmatrix}, \tag{EC.16}
\]

with eigenvalues

\[
\lambda_{1,2} = \phi'(0) \left(h_R - h_C \pm \sqrt{(h_R - h_C)^2 - 4(1-h_R-h_C)}\right). \tag{EC.17}
\]

\((1/2,1/2)\) is locally asymptotically stable if and only if both eigenvalues have negative real parts, i.e., \( h_R - h_C > 0 \) and \( h_C + h_R < 1 \). This implies Theorem 4(a).

- **Case (i)** \( h < 1/2, h_C < 1/2 \). From the above discussion, stability condition for the only fixed point \((1/2,1/2)\) is \( h_R - h_C > 0 \), because the second condition \( h_C + h_R < 1 \) is always satisfied in this case.

- **Case (ii)** \( h > 1/2, h_R \geq 1/2 \). The interior fixed point \((1/2,1/2)\) must be unstable since \( h_C + h_R > 1 \). ODE (6) has two boundary fixed points \((0, \frac{1}{2h_R})\) and \((1, 1 - \frac{1}{2h_R})\). The Jacobian matrix at \((0, \frac{1}{2h_R})\) is

\[
\begin{pmatrix}
-\phi(v_{c1}) & 0 \\
2\phi'(0)(1-h_R) & 2\phi'(0)h_R
\end{pmatrix}, \tag{EC.18}
\]

where \( v_{c1} = \frac{1-h_C-h_R}{h_R} < 0 \). The two eigenvalues of (EC.18) are \( \lambda_1 = -\phi(v_{c1}) < 0 \) and \( \lambda_2 = 2\phi'(0)h_R < 0 \). Hence \((0, \frac{1}{2h_R})\) is locally asymptotically stable. Similarly, \((1, 1 - \frac{1}{2h_R})\) is also stable.

- **Case (iii)** \( h > 1/2, h_C > 1/2, h_R < 1/2 \). The interior fixed point \((1/2,1/2)\) must be unstable since \( h_C + h_R > 1 \). ODE (6) has two boundary fixed points, \((0,1)\) and \((1,0)\). The Jacobian matrix at \((0,1)\) is

\[
\begin{pmatrix}
-\phi(v_{c1}) & 0 \\
0 & -\phi(v_{R2})
\end{pmatrix}, \tag{EC.19}
\]

where \( v_{c1} = 1-2h_C < 0 \) and \( v_{R2} = 2h_R - 1 < 0 \). The two eigenvalues of (EC.19) are \( \lambda_1 = -\phi(v_{c1}) < 0 \) and \( \lambda_2 = -\phi(v_{R2}) < 0 \). Hence \((0,1)\) is locally asymptotically stable. Similarly, \((1,0)\) is also stable.
• **Case (iv)** $h < 1/2, h_C > 1/2, h_R < 1/2$. The interior fixed point $(1/2, 1/2)$ must be unstable since $h_C > h_R$. ODE (6) has four boundary fixed points $(0,1), (1,0), (1/h-C,0)$ and $(1-1/2h_C,1)$. Similar to Case (iii), $(0,1)$ and $(1,0)$ are stable. The Jacobian matrix at $(1/h-C,0)$ is

\[
\begin{pmatrix}
-2\phi'(0)h_C & -2\phi'(0)(1-h_C) \\
0 & -\phi(v_{R1})
\end{pmatrix},
\]

where $v_{R1} = h_C + h_R^{-1} h_C < 0$. The two eigenvalues of (EC.20) are $\lambda_1 = -2\phi'(0)h_C > 0$ and $\lambda_2 = -\phi(v_{R1}) < 0$. Thus, $(1/h-C,0)$ is an unstable saddle point. Similarly, $(1-1/2h_C,1)$ is also a saddle point.

• **Case (v)** $h_C = 1/2, h_R < 1/2$. The interior fixed point $(1/2, 1/2)$ must be unstable since $h_C > h_R$. ODE (6) has two boundary fixed points $(0,1)$ and $(1,0)$. The Jacobian matrix at $(0,1)$ is

\[
\begin{pmatrix}
-2\phi'(0)h_C & -2\phi'(0)(1-h_C) \\
0 & -\phi(v_{R2})
\end{pmatrix},
\]

where $v_{R2} = 2h_R - 1 < 0$. The two eigenvalues of (EC.21) are $\lambda_1 = -2\phi'(0)h_C > 0$ and $\lambda_2 = -\phi(v_{R2}) < 0$. Thus, $(0,1)$ is an unstable saddle point. Similarly, $(1,0)$ is also a saddle point.

• **Case (vi)** $h = 1/2$. We will show in the next subsection that trajectories of ODE (6) converge to the line $y = \frac{1}{2(1-h_C)} - \frac{h_C}{1-h_C} x$. Thus, points on this line are neutrally stable.

From Case (ii) and Case (iii), we obtain Theorem 4(b).

**EC.5.3. Proof to Theorem 5 and Corollary 2**

On the $x-y$ plane, zero-isoclines for $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} = 0$ are respectively

\[
L_x : y = \frac{1}{2(1-h_C)} - \frac{h_C}{1-h_C}x;
\]
\[
L_y : y = \frac{1}{2h_R} - \frac{1-h_R}{h_R}x.
\]

Note that in the left parts of Figure 4, solid red lines are $L_x$ and dashed black ones are $L_y$. Points satisfying $\frac{dx}{dt} > 0(<0)$ are above (below) $L_x$ and those meeting $\frac{dy}{dt} < 0(>0)$ are above (below) $L_y$. Therefore the $x-y$ plane is divided into four regions (except for Case (vi), which is not demonstrated in Figure 4) and the two lines have a unique intersection $(1/2,1/2)$. For convenience, we denote the slopes of $L_x$ and $L_y$ by $l_x$ and $l_y$, respectively.

• **Case (i)** $h < 1/2, h_C < 1/2$. In this case, we have $l_x > -1$ and $l_x > l_y$. Intuitively, trajectories of ODE (6) move clockwise around $(1/2,1/2)$ and always start from the boundaries. If the interior fixed point $(1/2,1/2)$ is unstable, then ODE (6) must have a periodic solution (see Figure 4(i1)). If $(1/2,1/2)$ is locally stable, it may also be globally attractive (see Figure 4(i2)).

• **Case (ii)** $h > 1/2, h_R \geq 1/2$. In this case, we have $l_y > -1$ and $l_x < l_y$. The interior fixed point $(1/2,1/2)$ is unstable and two boundary fixed points $(0,1/h-R)$ and $(1,1-1/2h_R)$ are locally
asymptotically stable. In this case, existence of a periodic solution is impossible because the vector fields in the four regions cannot form a cycle (see Figure 4(ii)).

- **Case (iii)** \( h > 1/2, h_C > 1/2, h_R < 1/2 \). In this case, we have \( l_x < l_y < -1 \). The interior fixed point \((1/2, 1/2)\) is unstable and two boundary fixed points \((0, 1)\) and \((1, 0)\) are locally asymptotically stable. The phase portrait in this case looks like Case (ii), where ODE (6) has no periodic solution. (See Figure 4(iii))

- **Case (iv)** \( h < 1/2, h_C > 1/2, h_R < 1/2 \). In this case, we have \(-1 > l_x > l_y\). \((0, 1)\) and \((1, 0)\) are locally asymptotically stable, and \(\left(\frac{1}{2h_C}, 0\right)\) and \(\left(1 - \frac{1}{2h_C}, 1\right)\) are saddle points. The global dynamic behavior could be very complicated. ODE (6) may have two clockwise periodic solutions if the interior fixed point \((1/2, 1/2)\) is unstable (see Figure 4(iv)).

- **Case (v)** \( h_C = 1/2, h_R < 1/2 \). In this case, we have \(-1 = l_x > l_y\). \((0, 1)\) and \((1, 0)\) are saddle points. Similar to Case (i), trajectories of ODE (6) move clockwise around \((1/2, 1/2)\). If the interior fixed point \((1/2, 1/2)\) is unstable, ODE (6) must have a periodic solution (see Figure 4(v)). On the other hand, even if the interior fixed point is locally stable, ODE (6) may have two periodic solutions.

- **Case (vi)** \( h = 1/2 \). In this case, we have \( l_x = l_y \), i.e., the two zero-isoclines are identical, and all points on the line are fixed points. In this case, trajectories of ODE (6) converge to the line \( y = \frac{1}{2(1-h_C)} - \frac{h_C}{1-h_C} x \).

Overall, periodic solutions exist in Case (i) when \((1/2, 1/2)\) is unstable, i.e., \( h < 1/2, h_C < 1/2 \) and \( h_C > h_R \), and may exist in Case (iv) and Case (v), i.e., \( h < 1/2, h_C \geq 1/2 \) and \( h_R < 1/2 \). Furthermore, periodic solutions must rotate clockwise if exist. These imply Theorem 5(a) and Corollary 2. On the other hand, in Cases (ii) and Cases (iii), existence of a periodic solution is impossible and almost all trajectories converge to boundary fixed points. These yield Theorem 5(b).

**EC.5.4. Regular Graphs**

If a graph is regular, from Proposition 5, \( h < 1/2 \) if and only if \( h_C < f_C \). Thus, fixed points and their stabilities in regular graphs can be summarized in Table EC.1. The global dynamic behaviors of ODE(6) are shown in Figure EC.1.

**EC.6. Numerical Simulations**

Since the deriving of ODE (6), from SBRD, applies approximation techniques, a natural question is how good the approximation is. To answer this question, we do error analysis through two approaches. First, we compare the trajectories of ODE (6) and SBRD, as shown in Figure 4. Second, we calculate the differences between the fixed points of ODE (6) and SBRD. Note that although
Table EC.1  Fixed points of ODE (6) and their stabilities in regular graphs.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Fixed points</th>
<th>Stability conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) $h_C &lt; f_C$, $h_C &lt; 1/2$</td>
<td>$(1/2, 1/2)$</td>
<td>$f_C &lt; 1/2$</td>
</tr>
<tr>
<td>(ii) $h_C &gt; f_C$, $h_R \geq 1/2$</td>
<td>$(1/2, 1/2)$</td>
<td>unstable</td>
</tr>
<tr>
<td></td>
<td>$(0, 1)$, $(1, 0)$</td>
<td>stable</td>
</tr>
<tr>
<td>(iii) $h_C &gt; f_C$, $h_R &lt; 1/2$</td>
<td>$(1/2, 1/2)$</td>
<td>unstable</td>
</tr>
<tr>
<td></td>
<td>$(0, 1)$, $(1, 0)$</td>
<td>stable</td>
</tr>
<tr>
<td>(iv) $h_C &lt; f_C$, $h_R &gt; 1/2$</td>
<td>$(1/2, 1/2)$</td>
<td>unstable</td>
</tr>
<tr>
<td></td>
<td>$(0, 1)$, $(1, 0)$</td>
<td>stable</td>
</tr>
<tr>
<td></td>
<td>$(1/2h_C, 0)$, $(1 - 1/2h_C, 1)$</td>
<td>unstable</td>
</tr>
<tr>
<td>(v) $h_C &lt; f_C$, $h_C = 1/2$</td>
<td>$(1/2, 1/2)$</td>
<td>unstable</td>
</tr>
<tr>
<td></td>
<td>$(0, 1)$, $(1, 0)$</td>
<td>stable</td>
</tr>
<tr>
<td>(vi) $h_C = f_C$</td>
<td>$y = \frac{1}{2(1-h_C)} - \frac{h_C}{1-h_C} x$</td>
<td>neutral stable</td>
</tr>
</tbody>
</table>

Figure EC.1  Fixed points of ODE (6) for regular graphs. The curve separating areas (ii) and (iii) is $h_C = \frac{3}{2} - \frac{1}{2f_C}$. Fashion cycles emerge only in the gray region.

The fixed points of ODE (6) are easily computable, as shown in Table 1, those of SBRD, i.e. their absorbing states, are hard to compute precisely. To solve this problem, we recur to numerical simulations again.

**EC.6.1. Network Generation**

To guarantee the validity of our error analysis, we need to generate a wide class of fashion games with controllable parameters $f_C$, $h_C$ and $h_R$. Note that our proof to Proposition 2 is algorithmic. With the assistance of that algorithm, we can generate a fashion game with arbitrary combination of parameters. However, that class of fashion games are overly special. In this subsection, we introduce a variant of the Erdős-Renyi model. We did not consider more realistic small-world networks or scale-free networks. The key reason is that homophily indices are hard to control there (this is a good problem for future research).

Given the number $n$ of total nodes of the network, and the fraction $f_C$ of conformists, types of agents are randomly and independently assigned. Each agent has a probability of $f_C$ to be a
conformist, and $1 - f_C$ to be a rebel. The main trick is that we use three wiring probabilities, $p_{CC}$, $p_{CR}$ and $p_{RR}$ to tune the parameters $h_C$ and $h_R$. For each pair of conformists, the probability that they are connected is $p_{CC}$. For each pair of rebels, that probability is $p_{RR}$, and for any conformist and a rebel, it is $p_{CR}$. Note that in the Erdős-Renyi model, there is only one such probability. We are left to show how to decide the three probabilities.

Under this wiring process, the expected numbers of CC, CR and RR edges are respectively

$$k_{CC} = \binom{n f_C}{2} p_{CC}, \quad k_{CR} = n f_C (1 - f_C) p_{CR}, \quad k_{RR} = \binom{n(1 - f_C)}{2} p_{RR}.$$  

From Definition 1, we have

$$h_C = \frac{(n f_C - 1) p_{CC}}{(n f_C - 1) p_{CC} + n (1 - f_C) p_{CR}}, \quad h_R = \frac{(n - n f_C - 1) p_{RR}}{(n - n f_C - 1) p_{RR} + n f_C p_{CR}}.$$  

This gives

$$p_{CC} = \frac{np_{CR} h_C (1 - f_C)}{(1 - h_C) (n f_C - 1)}, \quad (EC.22)$$  

$$p_{RR} = \frac{np_{CR} h_R f_C}{(1 - h_R) (n - n f_C - 1)}. \quad (EC.23)$$

It can be checked that, for arbitrary $f_C \in \left(\frac{1}{n}, 1 - \frac{1}{n}\right)$ and $h_C, h_R \in [0, 1)$, there exists $p_{CR} \in [0, 1]$ such that $p_{CC}$ and $p_{RR}$ both fall into $[0, 1]$. Thus the above construction is valid.

**EC.6.2. Trajectory Comparison**

Our analysis in EC.5.3 is valid for any form of switching function $\phi(\cdot)$, as long as it is differentiable and decreasing for non-positive values. In this trajectory comparison, as demonstrated in Figure 4, we only consider a simple linear form, i.e. $\phi(v) = -v$ if $v < 0$ and $\phi(v) = 0$ if $v \geq 0$, in both dynamics.

In the simulations, the population consists of 200 conformists and 200 rebels (i.e., $f_C = 0.5$). The wiring probability $p_{CR}$ is taken as 0.05, which guarantees connectivity. For any pair of $h_C$ and $h_R$, $p_{CC}$ and $p_{RR}$ are derived from (EC.22) and (EC.23). Depending on the homophily indices, average degrees in different games vary from 10 to 30. In each simulation, SBRD starts from five different initial values $(0.5, 0.5), (0.5, 0.3), (0.5, 0.5), (0.5, 0.7)$ and $(0.5, 0.9)$. That is, each of the six right parts of Figures 4 is a combination of five trajectories. The simulation stops at the $10^5$-th time step.

As predicted by the deterministic dynamic ODE (6), in Figures 4(ii) and 4(iii), trajectories move clockwise around $(0.5, 0.5)$ and go always from boundaries. In Figures 4(ii) and 4(iii), the existence of a periodic solution is impossible and trajectories evolve to boundaries. In Figures 4(iv) and 4(v), $(0.5, 0.5)$ is unstable and trajectories converge to periodic solutions clockwise.
EC.6.3. Point Comparison

As described in the previous subsection, trajectories of SBRD and the corresponding ODE can only be qualitatively compared. For fixed points of ODE (6) and the (possibly approximate) absorbing states of the corresponding SBRD, luckily, their comparison can be done more quantitively and thus more precisely.

We run two groups of simulations. In the first group, the population consists of 50 conformists and 50 rebels, and in the second group, 200 conformists and 200 rebels. In each group of simulations, we generate 1000 fashion games with 100 different combinations of homophily indices \((h_C, h_R)\) (i.e. 10 games per \((h_C, h_R)\)), where \(h_C\) and \(h_R\) are both evenly sampled from 0 to 0.9 (i.e. each takes 10 values).

For each game, we run the SBRD for \(10^5\) time steps from 25 different initial points \((x, y)\) (so there are 25 trajectories in total for each game), where \(x\) and \(y\) are both evenly sampled from 0.1 to 0.9 (i.e. each takes 5 values). For each initial point, we use \((\bar{x}, \bar{y})\) to denote its (possibly approximate) fixed point: (i) If the action profile of the final step is a PNE, we let \((\bar{x}, \bar{y})\) be the population state derived by that PNE. (ii) Otherwise, we let \((\bar{x}, \bar{y})\) be the average of the last \(10^4\) population states derived by the corresponding action profiles.

On the other hand, for each initial point, we denote \((\hat{x}, \hat{y})\) as the limiting value of the corresponding trajectory of the ODE, when it does converge to a single value. If the trajectory converges to a limit cycle, we simply let \((\hat{x}, \hat{y}) = (1/2, 1/2)\), which is the time average of the limit cycle.

For each initial point, the absolute error, i.e. the difference between \((\bar{x}, \bar{y})\) and \((\hat{x}, \hat{y})\), is measured by \(\varepsilon = \max\{|\bar{x} - \hat{x}|, |\bar{y} - \hat{y}|\}\). Finally, we define the absolute error for each combination of homophily indices \((h_C, h_R)\) by the average of its 250 samples (10 games \(\times\) 25 initial values per game). The final results are shown in Figure 5.

Generally speaking, the absolute errors are fairly small. When \(n = 100\), its overall average value is 0.077 (Figure 5(a)), and when \(n = 400\), it drops down to 0.037 (Figure 5(b)). This matches very well the prediction of the previous studies on diffusion approximation: error drops by one half as the population size increases by four times (Traulsen et al., 2006). From Figure 5, we can also observe that error is large for initial points that are close to lines \(h_R + h_C = 1\), \(h_C = 0.5\) and \(h_R = 0.5\). A possible reason is that homophily indices on the three lines are bifurcation values of the ODE. Therefore, trajectories of the corresponding SBRD are very sensitive to stochastic factors.

We also calculate the absolute errors for two benchmark cases. In both cases, the average errors are much less than those of general cases. If the population consists of only conformists, the average error is 0.0419 when \(n = 100\), and is less than 0.0001 when \(n = 400\). On the other hand, if the population consists of only rebels, the average error is 0.0177 when \(n = 100\), and is 0.0055 when \(n = 400\).
EC.7. Examples

Figure EC.2  A fashion game that is strong conformist homophilic but not a potential game. Deviating orders are rather special: the right two players have privilege over the left two ones, i.e. whenever two players, one on the left and the other on the right, both have incentive to deviate (as in the lower two cases), the left one has to wait for the next round. In this example, the upper-left player never has chance to deviate. Under this special deviating order, BRD does not converge. Therefore, this fashion game is not a potential game. However, it is trivial to check that it is indeed strong conformist homophilic.

Figure EC.3  An example of convergence failure of SBRD. Conformists and rebels are represented by circles and triangles, respectively. Actions are indicated by colors: black for 1 and white for 0. Left: The two PNEs. Right: A trajectory of SBRD that does not converge.
Fixed points of ODE (6) may be inconsistent with PNEs. Conformists and rebels are represented by circles and triangles, respectively. Actions are indicated by colors: black for 1 and white for 0. Up: a fashion game whose ODE has a fixed point that is not consistent with any PNE. Down: a fashion game whose PNE is not consistent with any fixed point of its ODE. In the upper example, we can calculate that $(h_C, h_R) = (1/2, 0)$. Then $(x, y) = (1, 0)$ is a fixed point of ODE (6). However, the only corresponding action profile $(1, 1, 0, 0)$ is not a PNE (the central agent is unsatisfied). In the lower example, it can be computed that $(h_C, h_R) = (6/7, 0)$. Action profile $(1, 1, 0, 0, 1)$, as illustrated in the figure, is obviously a PNE. However, it can be checked that the corresponding population state $(1/2, 1)$ is not a fixed point of ODE (6).