I. INTRODUCTION

This document provides supplementary materials for the results presented in [1], which were not included in the manuscript due to space considerations. This document is available online at www.escience.cn/people/dshi/miscellaneous.html. The notations used below are consistent with those used in [1].

II. EVENT-BASED APPROXIMATE MMSE ESTIMATOR FOR THE DETERMINISTIC EVENT-TRIGGERING CONDITIONS

A. The deterministic innovation triggering condition

The recursive equations of the approximate MMSE estimator for multi-sensor systems are shown by Theorem 7 in [2]. For the single-sensor scenario, the recursive equations for the deterministic innovation triggering schedule can be simplified as

\[ \hat{x}_k^- = A\hat{x}_{k-1} + Bu_{k-1}, \quad (S.1) \]
\[ P_k^- = AP_{k-1}A^T + Q, \quad (S.2) \]
\[ \hat{x}_k = \hat{x}_k^- + \gamma_k K_k (y_k - \hat{y}_k^-) + (1 - \gamma_k) K_k \hat{z}_k, \]
\[ K_k = P_k^- C^T (CP_k^- C^T + R)^{-1}, \]
\[ P_k = (I - \vartheta_k K_k C)P_k^- \quad (S.3) \]

where \( \hat{z}_k \) and \( \vartheta_k \) are defined by

\[
\hat{z}_k = \left( \phi \left( \frac{-\delta}{Q_{\hat{z}_k^2}} \right) - \phi \left( \frac{\delta}{Q_{\hat{z}_k^2}} \right) \right) Q_{\hat{z}_k}^{1/2} = 0, \quad (S.4)
\]

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\[ \theta_k = \left[ \phi \left( \frac{-\delta}{Q^{1/2}_{1k}} \right) - \phi \left( \frac{\delta}{Q^{1/2}_{1k}} \right) \right]^2 - \frac{-\delta}{Q^{1/2}_{1k}} \phi \left( \frac{-\delta}{Q^{1/2}_{1k}} \right) - \frac{\delta}{Q^{1/2}_{1k}} \phi \left( \frac{\delta}{Q^{1/2}_{1k}} \right) = \frac{-\delta}{Q^{1/2}_{1k}} \phi \left( \frac{-\delta}{Q^{1/2}_{1k}} \right) - \frac{\delta}{Q^{1/2}_{1k}} \phi \left( \frac{\delta}{Q^{1/2}_{1k}} \right). \] (S.5)

where \( \phi(z) := \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2} z^2) \), \( Q_{zk} := CP_k^T C + R \) and \( Q(\cdot) \) denotes the standard Q-function [3].

**B. The deterministic “send-on-delta” condition**

For the deterministic “send-on-delta” schedule, the recursive equations of the corresponding estimator are the same with those of the estimator based on the deterministic innovation triggering schedule other than the definitions of \( \dot{z}_k \) and \( \theta_k \), which are defined by

\[ \dot{z}_k = \frac{\phi \left( \frac{y_{rk} - \hat{y}_k - \delta}{Q^{1/2}_{1k}} \right) - \phi \left( \frac{y_{rk} - \hat{y}_k + \delta}{Q^{1/2}_{1k}} \right)}{Q \left( \frac{-\delta}{Q^{1/2}_{1k}} \right) - Q \left( \frac{\delta}{Q^{1/2}_{1k}} \right)} Q^{1/2}_{zk}, \] (S.6)

\[ \theta_k = \left[ \phi \left( \frac{y_{rk} - \hat{y}_k - \delta}{Q^{1/2}_{1k}} \right) - \phi \left( \frac{y_{rk} - \hat{y}_k + \delta}{Q^{1/2}_{1k}} \right) \right]^2 - \frac{\phi \left( \frac{y_{rk} - \hat{y}_k - \delta}{Q^{1/2}_{1k}} \right)}{Q \left( \frac{-\delta}{Q^{1/2}_{1k}} \right) - Q \left( \frac{\delta}{Q^{1/2}_{1k}} \right)} \phi \left( \frac{y_{rk} - \hat{y}_k + \delta}{Q^{1/2}_{1k}} \right) \frac{-\delta}{Q^{1/2}_{1k}} \phi \left( \frac{y_{rk} - \hat{y}_k - \delta}{Q^{1/2}_{1k}} \right) - \frac{\delta}{Q^{1/2}_{1k}} \phi \left( \frac{y_{rk} - \hat{y}_k + \delta}{Q^{1/2}_{1k}} \right) = \frac{\phi \left( \frac{y_{rk} - \hat{y}_k - \delta}{Q^{1/2}_{1k}} \right)}{Q \left( \frac{-\delta}{Q^{1/2}_{1k}} \right) - Q \left( \frac{\delta}{Q^{1/2}_{1k}} \right)} \phi \left( \frac{y_{rk} - \hat{y}_k + \delta}{Q^{1/2}_{1k}} \right). \] (S.7)

**III. MMSE estimator for the stochastic send-on-delta event-triggerring condition**

In [1], an event-triggering strategy called “stochastic send-on-delta” schedule is considered. For this transmission protocol, the probability of the current measurement \( y_k \) transmitted to the estimator is subjected to an analogous Gaussian-type distribution with mean value \( y_{rk} \) and covariance \( Y^{-1} \), where the communication rate is tuned by the parameter \( Y \). This schedule is written as:

\[ \gamma_k := \begin{cases} 0, & \zeta_k \leq \varphi(y_k, y_{rk}), \\ 1, & \zeta_k > \varphi(y_k, y_{rk}), \end{cases} \] (S.8)

where

\[ \varphi(y_k, y_{rk}) = \exp\left[ -\frac{1}{2} (y_k - y_{rk})^T Y (y_k - y_{rk}) \right], \] (S.9)

The MMSE event-based estimator is derived in this section. Note that at time \( k \), \( y_{rk} \) is known by both the sensor scheduler and the remote estimator, and therefore it is treated as a constant in estimator design. To aid the derivations, we assume \( D = 0 \) in equation (2) of [1]; the case of \( D \neq 0 \) can be proved similarly. Note that \( D = 0 \) holds for the state space model of the MPDM system developed in [1, Section III-B]. We have the following result.

**Theorem 1:** For this schedule, \( x_k \) conditioned \( I_{k-1} \) is Gaussian distributed with mean \( \hat{x}_k \) and covariance \( P_{1k} \), and \( x_k \) conditioned on \( I_k \) is Gaussian distributed with mean \( \hat{x}_k \) and covariance \( P_k \), where \( \hat{x}_k \), \( x_k \) and \( P_{1k}, P_k \) satisfying the following recursive equations:
Time Update:

$$\hat{x}_k = A\hat{x}_{k-1} + Bu_{k-1},$$  \hspace{1cm} (S.10)

$$P^{-}_k = AP_{k-1}A^T + Q,$$  \hspace{1cm} (S.11)

Measurement Update:

$$K_k = P^{-}_k C^T [CP^{-}_k C^T + R + (1 - \gamma_k)Y^{-1}]^{-1},$$  \hspace{1cm} (S.12)

$$P_k = (I - K_k C) P^{-}_k,$$  \hspace{1cm} (S.13)

$$\hat{x}_k = \hat{x}_{k-1} + \gamma_k K_k(y_k - \hat{y}_k) + (1 - \gamma_k)K_k(y_{\tau_k} - \hat{y}_\tau_k),$$  \hspace{1cm} (S.14)

with initial condition

$$\hat{x}_{0} = 0, P^{-}_0 = \Sigma_0.$$  \hspace{1cm} (S.15)

Proof: Since $I_0 = \emptyset$, $x_0$ is Gaussian and (S.15) holds. We first consider the measurement update step. Assume that $x_k$ conditioned on $I_{k-1}$ is Gaussian with mean $\hat{x}_{k-1}$ and covariance $P^{-}_k$. We consider two cases depending on whether the estimator receives $y_k$.

1) If $\gamma_k = 0$, then the estimator does not receive $y_k$. The joint conditional probability density function (pdf) of $x_k$ and $y_k$ is written as

$$f(x_k, y_k | I_k) = f(x_k, y_k | \gamma_k = 0, I_{k-1}) \hspace{1cm} (S.16)$$

where the second equality follows from the Bayes' theorem and the last one holds since $\gamma_k$ depends only on $y_k$. By assumption, $f(x_k, y_k | I_{k-1})$ is a Gaussian distribution and

$$Pr(\gamma_k = 0 | y_k) = \exp\left[-\frac{1}{2}(y_k - y_{\tau_k})^T Y (y_k - y_{\tau_k})\right].$$  \hspace{1cm} (S.19)

In this way, we have

$$f(x_k, y_k | I_k) = \alpha_k \exp\left(-\frac{1}{2} \theta_k\right),$$  \hspace{1cm} (S.20)

with

$$\alpha_k = \frac{1}{Pr(\gamma_k = 0 | I_{k-1}) \sqrt{\det(\Phi_k)} (2\pi)^{m+n}}$$  \hspace{1cm} (S.21)

and

$$\theta_k = \begin{bmatrix} x_k - \hat{x}_{k-1} \\ y_k - \hat{y}_k \end{bmatrix}^T \Phi_k^{-1} \begin{bmatrix} x_k - \hat{x}_{k-1} \\ y_k - \hat{y}_k \end{bmatrix} + (y_k - y_{\tau_k})^TY (y_k - y_{\tau_k}),$$  \hspace{1cm} (S.22)

where $\Phi_k$ is the covariance of $[x_k^T, y_k^T]^T$ given $I_{k-1}$ and satisfies

$$\Phi_k := \begin{bmatrix} P^{-}_k & P^{-}_k C^T \\ CP^{-}_k & CP^{-}_k C^T + R \end{bmatrix},$$  \hspace{1cm} (S.23)
Based on some matrix calculations for (S.22) and the matrix inversion lemma, one has

\[ \Theta_k^{-1} = \begin{bmatrix} (P_k^-)^{-1} + C^T R^{-1} C & -C^T R^{-1} \\ -R^{-1} C & R^{-1} \end{bmatrix} \].

(S.24)

Based on some matrix calculations for (S.22) and the matrix inversion lemma, one has

\[ \theta_k = \begin{bmatrix} x_k - \bar{x}_k \\ y_k - \bar{y}_k \end{bmatrix}^T \Theta_k^{-1} \begin{bmatrix} x_k - \bar{x}_k \\ y_k - \bar{y}_k \end{bmatrix} + c_k, \]

(S.25)

where

\[ \bar{x}_k = \bar{x}_k + [C^T R^{-1} (R^{-1} + Y)^{-1} R^{-1} C - (P_k^-)^{-1} - C^T R^{-1} C]^{-1} C^T R^{-1} (R^{-1} + Y)^{-1} Y \bar{y}_k - y_{rk}, \]

(S.26)

\[ \bar{y}_k = \bar{y}_k - [R^{-1} + Y^{-1} C ((P_k^-)^{-1} + C^T R^{-1} C)^{-1} C^T R^{-1} Y]^{-1} Y (\bar{y}_k - y_{rk}), \]

(S.27)

\[ c_k = (\bar{y}_k - y_{rk})^T Y (R^{-1} + Y)^{-1} Y (\bar{y}_k - y_{rk}), \]

(S.28)

and

\[ \Theta_k = \begin{bmatrix} \Theta_{xx,k} & \Theta_{xy,k} \\ \Theta_{xy,k}^T & \Theta_{yy,k} \end{bmatrix}, \]

(S.29)

with

\[ \Theta_{xx,k} = P_k^- - P_k^- C^T (C P_k^- C^T + R + Y^{-1})^{-1} C P_k^-, \]

(S.30)

\[ \Theta_{xy,k} = P_k^- C^T [I + (C P_k^- C^T + R) Y]^{-1}, \]

(S.31)

\[ \Theta_{yy,k} = [(C P_k^- C^T + R)^{-1} + Y]^{-1}. \]

(S.32)

Thus,

\[ f(x_k, y_k | I_k) = \alpha_k \exp \left( -\frac{1}{2} \theta_k \right) \]

(S.33)

\[ = \alpha_k \exp \left( -\frac{1}{2} \begin{bmatrix} x_k - \bar{x}_k \\ y_k - \bar{y}_k \end{bmatrix}^T \Theta_k^{-1} \begin{bmatrix} x_k - \bar{x}_k \\ y_k - \bar{y}_k \end{bmatrix} + c_k \right) \]

(S.34)

\[ = \alpha_k \exp \left( -\frac{1}{2} c_k \right) \exp \left( \frac{1}{2} \begin{bmatrix} x_k - \bar{x}_k \\ y_k - \bar{y}_k \end{bmatrix}^T \Theta_k^{-1} \begin{bmatrix} x_k - \bar{x}_k \\ y_k - \bar{y}_k \end{bmatrix} \right). \]

(S.35)

Since \( f(x_k, y_k | I_k) \) is a pdf,

\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f(x_k, y_k | I_k) \, dx_k \, dy_k = 1, \]

(S.36)

which implies that

\[ \alpha_k \exp \left( -\frac{1}{2} c_k \right) = \frac{1}{\sqrt{\det(\Phi_k) (2\pi)^{m+n}}}. \]

(S.37)

As a result, \( x_k \) and \( y_k \) are jointly Gaussian given \( I_k \), which means that \( x_k \) is conditionally Gaussian with mean \( \bar{x}_k \) and covariance \( \Theta_{xx,k} \). Therefore, (S.13) and (S.14) hold when \( \gamma_k = 0 \).
2) If $\gamma_k = 1$, then the estimator receives $y_k$. Hence

$$f(x_k|\mathcal{I}_k) = f(x_k,y_k|\gamma_k = 1,\mathcal{I}_{k-1})$$

$$= \frac{\Pr(\gamma_k = 1|x_k,y_k,\mathcal{I}_{k-1})f(x_k|y_k,\mathcal{I}_{k-1})}{\Pr(\gamma_k = 1|y_k,\mathcal{I}_{k-1})}$$

$$= \frac{\Pr(\gamma_k = 1|y_k)f(x_k|y_k,\mathcal{I}_{k-1})}{\Pr(\gamma_k = 1|y_k)},$$

$$= f(x_k|y_k,\mathcal{I}_{k-1}).$$

(S.38)

(S.39)

(S.40)

(S.41)

The second equality is due to Bayes’ theorem and the third equality uses the conditional independence between $\gamma_k$ and $(\mathcal{I}_{k-1},x_k)$ given $y_k$. Since $y_k = Cx_k + v_k$ and $x_k,v_k$ are conditionally independently Gaussian distributed, $x_k$ and $y_k$ are conditionally jointly Gaussian which implies that $f(x_k|\mathcal{I}_k)$ is Gaussian. Following the standard Kalman filtering,

$$f(x_k|\mathcal{I}_k) \sim N(\hat{x}_k^- + K_k(y_k - \hat{y}_k^-), P_k^- - K_kCP_k^-).$$

(S.42)

Finally we consider the time update. Assume that $x_k$ conditioned on $\mathcal{I}_k$ is Gaussian distributed with mean $\hat{x}_k$ and covariance $P_k$.

$$f(x_{k+1}|\mathcal{I}_k) = f(Ax_k + Bu_k + w_k|\mathcal{I}_k).$$

(S.43)

Since $x_k$ and $w_k$ are conditionally mutually independent Gaussian and $u_k$ is a deterministic input, we have

$$f(x_{k+1}|\mathcal{I}_k) \sim N(A\hat{x}_k + Bu_k, AP_kA^T + Q),$$

(S.44)

which completes the proof.

From the standard estimation theory [4], $\hat{x}_k$ is the MMSE state estimate given the event-triggered measurement information $\mathcal{I}_k$.

REFERENCES


